GRADED LIE ALGEBRAS OF MAXIMAL CLASS IV

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Abstract. We describe the isomorphism classes of certain infinite-dimensional graded Lie algebras of maximal class, generated by an element of weight one and an element of weight two, over fields of odd characteristic.

1. Introduction

Let M be a Lie algebra over the field $\mathbf{F}$. Suppose M is nilpotent of nilpotency class c, so that c is the smallest number such that $M^{c+1} = 0$. If M has finite dimension $n \geq 2$, it is well-known that $c \leq n-1$. When $c = n-1$, M is said to be a Lie algebra of maximal class.

Consider the Lie powers M^i. Then M is of maximal class when the codimension of M^i is exactly i, for $i \leq c+1$. It is natural to extend the definition to an infinite-dimensional Lie algebra M by saying that M is of maximal class when the codimension of M^i is i for all i (see [6]).

One can grade M with respect to the filtration of the M^i: let

$$L_i = M^i/M^{i+1},$$

and consider

$$L = \bigoplus_{i=1}^{\infty} L_i. \tag{1}$$

There is a natural way of defining a Lie product on L, and the graded Lie algebra L has the following properties: $\dim(L_1) = 2$, $\dim(L_i) \leq 1$ for $i \geq 2$, and L is generated by L_1. Note that here too we allow all L_i to be non-zero, thereby including infinite-dimensional algebras. A graded Lie algebra L satisfying these conditions is called a graded Lie algebra of maximal class in [2, 3, 5]. However, this definition does not capture all possibilities. One of the other possibilities for a graded Lie algebra $L = \bigoplus_{i=1}^{\infty} L_i$ to be of maximal class is to have $\dim(L_i) \leq 1$ for all $i \geq 1$, with L generated by L_1 and L_2. We call a graded Lie algebra of this form an algebra of type 2, whereas we refer to a graded Lie algebra of maximal class in the sense of [2, 3, 5] as an algebra of type 1.

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In studying algebras of type 2, we will mainly deal with the infinite dimensional ones (as in [6, 2, 3, 5]). However, our arguments also provide fairly complete information about finite dimensional algebras.

If the characteristic of the underlying field $\mathbf{F}$ is zero, it is well-known that there is only one infinite dimensional algebra of type 1. This is the algebra

$$a = \left\langle\, x, y : [yx^i y] = 0, \text{ for all } i \geq 1 \,\right\rangle, \tag{2}$$

where x and y have weight 1. The ideal generated by y is an abelian maximal ideal here. However, if $\mathbf{F}$ has prime characteristic p there are uncountably many algebras of type 1 [6, 2]; these algebras were classified in [3, 5].

Over a field $\mathbf{F}$ of characteristic zero there are three infinite-dimensional algebras of type 2 [7, 4], called m, m_2 and W, and these are defined over the integers. The first one is a close analogue to a. It is given as

$$m = \left\langle\, e_1, e_2 : [e_2 e_1^i e_2] = 0, \text{ for all } i \geq 1 \,\right\rangle, \tag{3}$$

where e_1 has weight 1 and e_2 has weight 2. The ideal generated by e_2 is an abelian maximal ideal here. The second one is defined as

$$\begin{aligned} m_2 = \Big\langle\, e_i, i \geq 1 : [e_i e_1] &= e_{i+1}, \text{ for all } i \geq 2, \\ [e_i e_2] &= e_{i+2}, \text{ for all } i \geq 3 \\ [e_i e_j] &= 0, \text{ for all } i, j \geq 3 \Big\rangle, \end{aligned} \tag{4}$$

where e_i has weight i. Here $m_2^2 = \langle\, e_i : i \geq 3 \,\rangle$ is a maximal abelian ideal. The third algebra is the positive part of the Witt algebra:

$$W = \langle\, e_i, i \geq 1 : [e_i e_j] = (i-j) e_{i+j} \,\rangle,$$

and is not soluble.

When one considers these algebras over a field $\mathbf{F}$ of prime characteristic $p > 2$, m and m_2 give algebras of type 2, but W does not.

We will show in the next section that there is a natural way to obtain an algebra of type 2 from an uncovered algebra of type 1. (See the next section for the relevant definition.) In particular, m arises from a in this way. We will show that for prime characteristic $p > 2$ the algebras of type 2 consist of

- algebras arising in this natural way from algebras of type 1,
- m_2,
- one further family of soluble algebras,
- in the case $p = 3$, one additional family of soluble algebras.

This yields a classification of algebras of type 2 over fields of characteristic $p > 2$. We believe the case of characteristic two to be considerably more complicated.

2. Preliminaries

Let L be an *infinite-dimensional* Lie algebra over a field $\mathbf{F}$ that is graded over the positive integers:

$$L = \bigoplus_{i=1}^{\infty} L_i. \tag{5}$$

If $\dim(L_1) = 2$, $\dim(L_i) = 1$ for $i > 1$, and L is generated by L_1, we say that L is *an algebra of type* 1. These are the algebras that are called algebras of maximal class in [2, 3, 5]. In these papers these algebras are classified over fields of prime characteristic p.

As mentioned in the Introduction, over a field of characteristic zero there is only one isomorphism class of algebras of type 1. This is the algebra a of (2) generated by two elements x and y of weight 1, subject to the relations $[yx^iy] = 0$, for all $i \geq 1$. This algebra is metabelian, and the graded maximal ideal containing y is abelian. Here we use the notation

$$[yx^iy] = [y\underbrace{x \dots x}_{i}y].$$

If in the algebra (5) we have $\dim(L_i) = 1$ for all $i \geq 1$, and if L is generated by L_1 and L_2, we say that L is *an algebra of type* 2. Choose non-zero elements $e_1 \in L_1$ and $e_2 \in L_2$. Since L is of maximal class, for each $i \geq 2$ we have $[L_iL_1] = L_{i+1}$. Therefore we can recursively define $e_{i+1} = [e_ie_1]$, for $i \geq 2$, and we have $L_i = \langle\, e_i \,\rangle$ for all i. We keep this notation fixed for the rest of the paper, allowing ourselves to rescale e_2 when needed.

In [2, 3], to which we refer the reader for all details, a theory of *constituents* has been developed for algebras of type 1 over fields of positive characteristic p. If L is such an algebra, define its *i-th two-step centralizer* as

$$C_i = C_{L_1}(L_i) = \{\, v \in L_1 : [uv] = 0 \text{ for } u \in L_i \,\},$$

for $i > 1$. Each C_i is a one-dimensional subspace of L_1. A special role is played by the *first two-step centralizer* C_2. In fact, the sequence of the two-step centralizers consists of patterns, called constituents, of the following type

$$C_i \neq C_2, \quad C_{i+1} = C_{i+2} = \cdots = C_{i+l} = C_2, \quad C_{i+l+1} \neq C_2.$$

Here l is called the *length of the constituent*. (We are following the definition of [3], which differs from that of [2].) The first constituent requires a special treatment: its length is defined as the smallest f such that $C_f \neq C_2$, and turns out to be of the form $f = 2q$, where $q = p^h$, for some h. It is proved in [2] that if the first constituent has length $2q$, then the constituents of L can have lengths of the form

$$2q, \qquad \text{or} \qquad 2q - p^t, \quad \text{for } 0 \leq t \leq h.$$

An algebra of type 1 is said to be *uncovered* if the union of the C_i does not exhaust all of M_1. It is proved in [2] that over any field of positive characteristic there are uncontably many uncovered algebras of type 1. (On the other hand, if the field is at most countable, there are algebras of type 1 that are not uncovered.) If $M = \bigoplus_{i=1}^{\infty} M_i$ is uncovered, there is an element $z \in M_1$ such that

$$[M_iz] = M_{i+1} \text{ for all } i \geq 1. \tag{6}$$

We consider the maximal graded subalgebra

$$L = \langle\, z \,\rangle \oplus \bigoplus_{i \geq 2} M_i$$

of M. Because of (6), L is an algebra of type 2. In addition, the algebra L inherits some kind of constituent pattern from M, as we will see in the following. From now on we will assume $p > 2$.

If we apply this procedure to the unique algebra $M = a$ of (2) of type 1 in characteristic zero, which is clearly uncovered, we get the algebra L of type 2 generated by an element e_1 of weight one and an element e_2 of weight two subject to the relations $[e_2 e_1^i e_2] = 0$, for all $i \geq 1$. This is the algebra m of (3).

In positive characteristic, note first of all that in L we may take $e_1 = z$, $e_2 = [yz]$ where $0 \neq y \in C_2$, and take $e_k = [e_{k-1} e_1]$ for $k > 2$. Suppose that in M we have a segment of the sequence of two-step centralizers of the form

$$C_2 = C_{n-2} = C_{n-1}, C_n = \langle\, y + \lambda z \,\rangle \neq C_2, C_{n+1} = C_{n+2} = C_2,$$

so that $\lambda \neq 0$. Note that the first constituent has length $2q \geq 6$ so that, in particular,

$$[e_3 e_2] = [[yzz][yz]] = [yzzyz] - [yzzzy] = 0.$$

We have

$$\begin{aligned}
[e_{n-1} e_2] &= [e_{n-1}[yz]] \\
&= [e_{n-1} yz] - [e_{n-1} zy] \\
&= -[e_{n-1} zy] && \text{as } C_{n-1} = C_2 = \langle\, y \,\rangle \\
&= -[e_{n-1} e_1 y] \\
&= -[e_n y] \\
&= [e_n, \lambda z] - [e_n, y + \lambda z] \\
&= \lambda e_{n+1}.
\end{aligned}$$

Similarly

$$\begin{aligned}
[e_n e_2] &= [e_n[yz]] \\
&= [e_n yz] - [e_n zy] \\
&= [e_n yz] && \text{as } C_{n+1} = C_2 = \langle\, y \,\rangle \\
&= [e_n y e_1] \\
&= [e_n, -\lambda z, e_1] + [e_n, y + \lambda z, e_1] \\
&= -\lambda e_{n+2}.
\end{aligned}$$

Finally

$$[e_{n+1} e_2] = 0$$

as $C_{n+1} = C_{n+2} = C_2$.

In view of this, we introduce a definition of constituents for algebras of type 2 that is compatible with the definition for algebras of type 1. Let L be an arbitrary algebra of type 2. If $[e_3 e_2] = [e_2 e_1 e_2] \neq 0$, we have no theory of constituents for L. Algebras of this type are dealt with in Section 3 and Section 7. If $[e_2 e_1 e_2] = 0$,

and for some n we have $[e_{n-1}e_2] = 0$, but $[e_n e_2] = \lambda e_{n+2} \neq 0$, for some $\lambda \neq 0$, then

$$\begin{aligned} 0 &= [e_{n-1}[e_2 e_1 e_2]] \\ &= -[e_{n-1}e_1e_2e_2] + 2[e_{n-1}e_2e_1e_2] - [e_{n-1}e_2e_2e_1] \\ &= -[e_{n-1}e_1e_2e_2] \\ &= -[e_n e_2 e_2] \\ &= -\lambda[e_{n+2}e_2], \end{aligned}$$

so that $[e_{n+2}e_2] = 0$. We are therefore led to the following definition. Let L be an algebra of type 1 in which $[e_2e_1e_2] = 0$. Suppose there are integers m, n such that

$$\begin{aligned} &[e_{m-1}e_2] = 0, \\ &[e_m e_2] = \eta e_{m+2}, && \text{with } \eta \neq 0, \\ &[e_{m+1}e_2] = \vartheta e_{m+2}, \\ &[e_{m+2}e_2] = \cdots = [e_{n-1}e_2] = 0, \\ &[e_n e_2] = \lambda e_{n+2}, && \text{with } \lambda \neq 0, \\ &[e_{n+1}e_2] = \mu e_{n+3}. \end{aligned}$$

We call this pattern a *constituent of length* $l = n - m$ *and type* (λ, μ). Note that ϑ and μ might well be zero.

Here, too, the first constituent requires an ad hoc treatment. If in the algebra L one has $[e_2e_1e_2] = 0$, and n is the smallest integer greater than 1 such that $[e_n e_2] \neq 0$, we say that the first constituent has length $n + 1$. If there is no such n, then L is isomorphic to the algebra m above.

We will see in Section 4 that the first constituent of an algebra of type 2 can have length $q + 1$ or $2q$, where q is a power of the characteristic of the underlying field. If the first constituent has length $2q$, we will see in Section 5 that L comes from an algebra of type 1 via the procedure described above. If the first constituent has length $q + 1$, we will see in Sections 6–9 that we obtain one soluble algebra of type 2 for $q > 3$, and a family of soluble algebras for $q = 3$.

We have just seen that an algebra of type 2 that comes from an algebra of type 1 has constituents of type $(\lambda, -\lambda)$. We now prove that the converse also holds.

Suppose all constituents of the algebra L of type 2 are of type $(\lambda, -\lambda)$. Consider the following partial linear map

$$\begin{cases} e_1 \mapsto -e_2 \\ e_2 \mapsto 0. \end{cases}$$

We show that we can extend this to a unique derivation D of weight 1 on the whole of L. In the extension M of L by D, we have $[De_1] = -e_1D = e_2$. Thus M is generated by the elements e_1 and D of weight 1, and it is an uncovered algebra of type 1.

We begin with $e_3D = [e_2e_1]D = [e_2D, e_1] + [e_2, e_1D] = 0$. Suppose now we come to the end of a constituent in L, so that we have

$$[e_{i-2}e_2] = 0, [e_{i-1}e_2] = \lambda e_{i+1}, [e_ie_2] = -\lambda e_{i+2}.$$

We have so far, proceeding by induction, $e_{i-2}D = 0$. Now

$$e_{i-1}D = [e_{i-2}e_1]D = [e_{i-2}D, e_1] + [e_{i-2}, e_1D] = -[e_{i-2}e_2] = 0.$$

Then

$$e_iD = [e_{i-1}e_1]D = [e_{i-1}D, e_1] + [e_{i-1}, e_1D] = -[e_{i-1}e_2] = -\lambda e_{i+1},$$

$$e_{i+1}D = [e_ie_1]D = [e_iD, e_1] + [e_i, e_1D] = -\lambda[e_{i+1}e_1] - [e_ie_2] = 0,$$

and

$$e_{i+2}D = [e_{i+1}e_1]D = [e_{i+1}D, e_1] + [e_{i+1}, e_1D] = -[e_{i+1}e_2] = 0,$$

so that we can continue by induction.

This definition of D is compatible with the relations $[e_{i-2}, e_2] = 0$, $[e_{i-1}, e_2] = -\lambda e_{i+1}$, $[e_i, e_2] = \lambda e_{i+2}$, $[e_{i+1}, e_2] = 0$. This is clear for all but the third one. For this we have

$$[e_iD, e_2] + [e_i, e_2D] = -\lambda[e_{i+1}, e_2] = 0 = e_{i+2}D.$$

In [2] a device for studying algebras of type 1 called *deflation* has been introduced. We now show that this can be applied also to algebras of type 2, and the result will be an algebra of type 1. This is useful in simplifying some proofs later on.

Let L be an algebra of type 2 as in (5). Consider its subalgebra

$$S = \bigoplus_{i=1}^{\infty} L_{ip}.$$

Grade S by assigning weight i to L_{ip}. Now S admits the derivation $D = \mathrm{ad}(e_1)^p$ which, in the new grading, has weight 1. We have

$$L_{ip}\,\mathrm{ad}(e_1)^p = [L_{ip}e_1^p] = L_{(i+1)p}.$$

It follows that the extension of S by D is a graded Lie algebra of maximal class, and it is generated by the two elements e_p and D of weight 1. Therefore it is an algebra of type 1.

In this section we have used several times the Jacobi identity $[z[yx]] = [zyx] - [zxy]$, and its consequence

$$[z[yx^n]] = \sum_{i=0}^{n}(-1)^i\binom{n}{i}[zx^iyx^{n-i}].$$

In such a formula, to evaluate binomial coefficients modulo a prime we will make use of Lucas' theorem, in the following form. Suppose a, b are non-negative integers, and $q > 1$ is a power of a prime p. Write $a = a_0 + a_1q$, and $b = b_0 + b_1q$, where the a_i and b_i are non-negative integers, and $a_0, b_0 < q$. Then

$$\binom{a}{b} \equiv \binom{a_0}{b_0} \cdot \binom{a_1}{b_1} \pmod{p}.$$

3. Characterizing m_2

In this section we start dealing with algebras of type 2 that do not admit a theory of constituents, that is, in which $[e_3, e_2] \neq 0$ We may thus assume without loss of generality $[e_3, e_2] = e_5$. We obtain

$$\begin{aligned} 0 &= [e_3 e_3] \\ &= [e_3 [e_2 e_1]] \\ &= [e_3 e_2 e_1] - [e_3 e_1 e_2] \\ &= e_6 - [e_4 e_2]. \end{aligned}$$

Suppose that

$$[e_i, e_1] = e_{i+1} \text{ for } i > 1,$$

$$[e_3, e_2] = e_5,\ [e_4, e_2] = e_6,\ [e_5, e_2] = ae_7,\ [e_6, e_2] = be_8,$$

$$[e_7, e_2] = ce_9,\ [e_8, e_2] = de_{10},\ [e_9, e_2] = fe_{11},\ [e_{10}, e_2] = ge_{12}.$$

Here a, b, c, d, f, g, are parameters.

$[[e_2, e_1, e_1], [e_2, e_1, e_1]] = 0$ gives $1 - 2a + b = 0$, so $b = -1 + 2a$.

$[[e_2, e_1, e_1, e_1], [e_2, e_1, e_1, e_1]] = 0$ gives $a - 3b + 3c - d = 0$.

Now $a - 3b + 3c - d = -5a + 3 + 3c - d$, and so $d = 3 - 5a + 3c$.

$[[e_2, e_1, e_1, e_1, e_1], [e_2, e_1, e_1, e_1, e_1]] = 0$ gives $b - 4c + 6d - 4f + g = 0$.

$$b - 4c + 6d - 4f + g = 17 - 28a + 14c - 4f + g,$$

so $g = -17 + 28a - 14c + 4f$.

Note that $[e_2, e_1, e_1, e_1] = -[e_1, e_2, e_2]$.

$[[e_2, e_1, e_1, e_1], [e_1, e_2, e_2]] = 0$ gives $bd - 2ad + ac = 0$.

$$\begin{aligned} &bd - 2ad + ac \\ &= (-1 + 2a)(3 - 5a + 3c) - 2a(3 - 5a + 3c) + ac \\ &= -3 + 5a - 3c + ac, \end{aligned}$$

so either $a = 3$ (which gives $12 = 0$), or $c = \dfrac{3 - 5a}{a - 3}$.

$$[[e_2, e_1, e_1, e_1, e_1], [e_2, e_1, e_1, e_1]] + [[e_2, e_1, e_1, e_1, e_1], [e_1, e_2, e_2]] = 0$$

gives

$$b - 3c + 3d - f + cf - 2bf + bd = 0.$$

$$\begin{aligned} b - 3c + 3d - f + cf - 2bf + bd &= \\ &= 8 - 13a + 6\frac{3 - 5a}{a - 3} - f + \frac{3 - 5a}{a - 3} f \\ &\quad - 2(-1 + 2a) f + (-1 + 2a)\left(3 - 5a + 3\frac{3 - 5a}{a - 3}\right) \\ &= -2\frac{-7a + 3 + a^2 - 4fa + 2fa^2 + 5a^3}{a - 3}. \end{aligned}$$

So provided the characteristic is not 2, and provided $a \neq 3$,
$$-7a + 3 + a^2 + 5a^3 + (2a^2 - 4a)f = 0.$$

$$[[e_2, e_1, e_1, e_1, e_1, e_1], [e_2, e_1, e_1, e_1]] + [[e_2, e_1, e_1, e_1, e_1, e_1], [e_1, e_2, e_2]] = 0$$
gives
$$c - 3d + 3f - g + dg - 2cg + cf = 0.$$
$$\begin{aligned} &c - 3d + 3f - g + dg - 2cg + cf \\ &= -2\frac{18 - 27f - 123a + 280a^2 + 78fa - 53fa^2 - 253a^3 + 10fa^3 + 70a^4}{(a-3)^2}. \end{aligned}$$
So provided the characteristic is not 2, and provided $a \neq 3$,
$$18 - 123a + 280a^2 - 253a^3 + 70a^4 + (-27 + 78a - 53a^2 + 10a^3)f = 0.$$
Combining these two equations we obtain
$$\begin{aligned} &(-27 + 78a - 53a^2 + 10a^3)(-7a + 3 + a^2 + 5a^3) - \\ &\qquad - (2a^2 - 4a)(18 - 123a + 280a^2 - 253a^3 + 70a^4) = 0 \end{aligned}$$
Expanding, we obtain
$$\begin{aligned} 0 &= 495a - 81 - 1260a^2 + 1710a^3 - 1305a^4 + 531a^5 - 90a^6 \\ &= -9 \times (10a - 9)(a - 1)^5. \end{aligned}$$
So if the characteristic is not 2 or 3 or 5 then $a = 1$ or $a = \frac{9}{10}$. If the characteristic is 5 then $a = 1$. The cases when the characteristic is 2 or 3 have to be dealt with separately. We deal with the latter in Section 7.

When $a = \frac{9}{10}$, it is proved in [1] that the algebras one obtains are quotients of a certain central extension of the positive part of the infinite-dimensional Witt algebra. In any case, there are no infinite-dimensional algebras of maximal class here.

The choice $a = 1$ uniquely determines the following metabelian Lie algebra [4, 7]:
$$\begin{aligned} m_2 = \langle e_i, i \geq 1 : {} & [e_i e_1] = e_{i+1}, \text{ for } i \geq 2, \\ & [e_i e_2] = e_{i+2}, \text{ for } i \geq 3, \\ & [e_i e_j] = 0, \text{ for } i, j \geq 3 \ \rangle. \end{aligned}$$
Note that $\mathrm{ad}(e_2)$ is the square of $\mathrm{ad}(e_1)$ on L^2.

In fact, we have to show that m_2 has the following presentation:
$$\langle\, e_1, e_2 : [e_2 e_1 e_2] = [e_2 e_1^3], [e_2 e_1^3 e_2] = [e_2 e_1^5] \,\rangle.$$
We use the notation $[e_i e_1] = e_{i+1}$, so that the two defining relations can be rewritten as $[e_3 e_2] = e_5$ and $[e_5 e_2] = e_7$. We have already seen that the first one implies $[e_4 e_2] = e_6$. Suppose now we have proved
$$[e_3 e_2] = e_5, [e_4 e_2] = e_6, [e_5 e_2] = e_7, \ldots, [e_{n-1} e_2] = e_{n+1},$$

for some $n > 5$, and want to prove $[e_n e_2] = e_{n+2}$. We work out the expansion

$$\begin{aligned}
0 &= [e_{n-3}, [e_3 e_2] - e_5] = [e_{n-3}, [e_2 e_1 e_2] - [e_2 e_1^3]] \\
&= [e_{n-3}[e_2 e_1] e_2] \\
&\quad - [e_{n-3} e_2 [e_2 e_1]] \\
&\quad - [e_{n-3}[e_2 e_1^3]] \\
&= (1-1)[e_n e_2] \\
&\quad - e_{n+2} + [e_n e_2] \\
&\quad - (1-3+3) e_{n+2} + [e_n e_2] \\
&= 2[e_n e_2] - 2e_{n+2}.
\end{aligned}$$

Note that this does not work for $n = 5$. From this it is straightforward to see that the algebra is metabelian, and thus is isomorphic to m_2. In fact we have for $i, j \geq 3$

$$\begin{aligned}
[e_i e_j] &= [e_i [e_2 e_1^{j-2}]] \\
&= \sum_{k=0}^{j-2} (-1)^k \binom{j-2}{k} [e_i e_1^k e_2 e_1^{j-2-k}] \\
&= \left(\sum_{k=0}^{j-2} (-1)^k \binom{j-2}{k} \right) \cdot e_{i+j} \\
&= 0.
\end{aligned}$$

4. The length of the first constituent

Suppose now L is an algebra of type 2 over a field of positive characteristic p. Suppose L admits a theory of constituents. Therefore $[e_3 e_2] = [e_2 e_1 e_2] = 0$. If $[e_i e_2] = 0$ for all $i \geq 3$, then L is isomorphic to m of (3). Suppose thus there is an $n > 3$ such that $[e_3 e_2] = [e_4 e_2] = \cdots = [e_{n-2} e_2] = 0$, but $[e_{n-1} e_2] \neq 0$. We intend to show that n, the length of the first constituent, can only assume the values

$$\begin{cases} 2q, & \text{for some power } q \text{ of } p \text{, or} \\ q+1, & \text{for some power } q > 3 \text{ of } p. \end{cases}$$

We may assume, rescaling e_2, that $[e_{n-1} e_2] = e_{n+1}$. We first prove that n is even, with a simple argument similar to one of [2]. In fact, if $n = 2k - 1$ is odd, we have

$$\begin{aligned}
0 &= [e_k e_k] = [e_k [e_2 e_1^{k-2}]] \\
&= \sum_{i=0}^{k-2} (-1)^i \binom{k-2}{i} [e_k e_1^i e_2 e_1^{k-2-i}] \\
&= \sum_{i=0}^{k-2} (-1)^i \binom{k-2}{i} [e_{k+i} e_2 e_1^{k-2-i}] \approx [e_{n-1} e_2] = e_{n+1},
\end{aligned}$$

a contradiction. Here and in the following we write $a \approx b$ to mean that a is either b or $-b$. Write $n = 2k$. We aim at proving that the only possible values for k are q and $(q+1)/2$.

We first compute

$$\begin{aligned} 0 &= [e_{k+1} e_{k+1}] = [e_{k+1}[e_2 e_1^{k-1}]] \\ &\approx \binom{k-1}{k-2}[e_{2k-1} e_2 e_1] - \binom{k-1}{k-1}[e_{2k} e_2] \\ &= (k-1) e_{2k+2} - [e_{2k} e_2], \end{aligned}$$

to show

$$[e_n e_2] = (k-1) e_{n+2}.$$

We now have

$$\begin{aligned} 0 &= [e_{n-2}[e_2 e_1 e_2]] \\ &\approx [e_{n-2} e_1 e_2 e_2] \\ &= [e_{n+1} e_2]. \end{aligned} \tag{7}$$

Further,

$$\begin{aligned} 0 &= [e_{n-1}[e_2 e_1 e_2]] \\ &\approx [e_{n-1} e_1 e_2 e_2] - 2[e_{n-1} e_2 e_1 e_2] + [e_{n-1} e_2 e_2 e_1] \\ &= (k-1-2)[e_{n+2} e_2]. \end{aligned} \tag{8}$$

This shows that $[e_{n+2} e_2] = 0$, except when $k \equiv 3 \pmod{p}$.

Suppose first we have $n = 6$, or $k = 3$. We have here

$$[e_5 e_2] = e_7, \quad [e_6 e_2] = 2e_8, \quad [e_7 e_2] = 0.$$

We want to show that $p = 3$ or 5 here, so that this fits into the $n = 2q$ or $q + 1$ pattern above. Suppose $p > 5$. We compute

$$\begin{aligned} 0 &= [e_5[e_2 e_1^3]] \\ &= [e_5 e_2 e_1^3] - 3[e_6 e_2 e_1^2] + 3[e_7 e_2 e_1] - [e_8 e_2] \\ &= -5 e_{10} - [e_8 e_2], \end{aligned}$$

so that $[e_8 e_2] = -5 e_{10}$.

$$\begin{aligned} 0 &= [e_6[e_1 e_2 e_2]] \\ &= [e_7 e_2 e_2] - 2[e_6 e_2 e_1 e_2] + [e_6 e_2 e_2 e_1] \\ &= -4[e_9 e_2] + 2(-5) e_{11}, \end{aligned}$$

so that

$$[e_9 e_2] = -\frac{5}{2} e_{11}.$$

Finally

$$\begin{aligned}
0 &= [e_6[e_2e_1^4]] \\
&= [e_6e_2e_1^4] - 4[e_7e_2e_1^3] + 6[e_8e_2e_1^2] - 4[e_9e_2e_1] + [e_{10}e_2] \\
&= 18e_{12} + [e_{10}e_2]
\end{aligned}$$

and

$$\begin{aligned}
0 &= [e_7[e_1e_2e_2]] \\
&= [e_8e_2e_2] - 2[e_7e_2e_1e_2] + [e_7e_2e_2e_1] \\
&= -5[e_{10}e_2]
\end{aligned}$$

yield $e_{12} = 0$, a contradiction.

Suppose then $k > 3$, that is, $n > 6$. We have thus $[e_5e_2] = 0$, so that

$$\begin{aligned}
0 &= [e_{n-3}[e_5e_2]] = [e_{n-3}[e_2e_1e_1e_1e_2]] \\
&= [e_{n-3}[e_2e_1e_1e_1]e_2]] \\
&= 3[e_{n-3}e_1e_1e_2e_1e_2] - [e_{n-3}e_1e_1e_1e_2e_2] \\
&= (3 - (k-1))[e_{n+2}e_2].
\end{aligned}$$

This shows that $[e_{n+2}e_2] = 0$, except when $k \equiv 4 \pmod{p}$, which was covered by (8).

To find out what the possible values of k are, we compute

$$\begin{aligned}
0 &= [e_{k+2}e_{k+2}] = [e_{k+2}[e_2e_1^k]] \\
&\approx \binom{k}{k-3}[e_{2k-1}e_2e_1e_1e_1] - \binom{k}{k-2}[e_{2k}e_2e_1e_1]
\end{aligned}$$

which yields

$$0 = \left(\binom{k}{3} - \binom{k}{2}(k-1)\right) e_{2k+4} = \frac{k(k-1)(-2k+1)}{6}\, e_{2k+4}.$$

This shows that the only possibilities for k are

$$k \in \left\{0, 1, \frac{1}{2}\right\} \pmod{p},$$

for $p > 3$, whereas for $p = 3$ one has

$$k \in \left\{0, 1, \frac{1}{2}\right\} \pmod{9}.$$

When $k \equiv 0 \pmod{p}$, we show that $k = q$, a power of p. (The case when $p = 3$ is not special here, as we have already dealt with $k = 3$ for $p = 3$ above.) This we do by exploiting the deflation procedure, as described in Section 2. Suppose in fact $k = qm$, with q a power of p, and $m \not\equiv 0 \pmod{p}$. Thus $n = 2qm$ here. We have $[e_{n-1}e_2] = e_{n+1}$ and $[e_ne_2] = -e_{n+2}$. We have also proved in (7) that $[e_{n+1}e_2] = 0$. We first extend this to

$$[e_{n+1}e_2] = [e_{n+2}e_2] = \cdots = [e_{n+p-2}e_2] = 0.$$

We proceed by induction on l, for $1 < l \le p-2$:

$$\begin{aligned} 0 &= [e_{n-1}[e_2 e_1^{l-1} e_2]] \\ &= [e_{n-1}[e_2 e_1^{l-1}]e_2] - [e_{n-1} e_2 [e_2 e_1^{l-1}]] \\ &= [e_{n-1} e_2 e_1^{l-1} e_2] - (l-1)[e_{n-1} e_1 e_2 e_1^{l-2} e_2] - (-1)^{l-1}[e_{n-1} e_2 e_1^{l-1} e_2] \\ &= (1 + l - 1 - (-1)^{l-1}) \cdot [e_{n+l} e_2]. \end{aligned} \tag{9}$$

Now

$$1 + l - 1 - (-1)^{l-1} = l + (-1)^l = \begin{cases} l-1 & \text{when } l \text{ is odd,} \\ l+1 & \text{when } l \text{ is even.} \end{cases}$$

In any case the coefficient of $[e_{n+l} e_2]$ is less than p for $l < p-1$, so that it is non-zero.

In the deflated algebra, we thus have

$$[e_{2qm-p} e_p] = [e_{2qm-p}[e_2 e_1^{p-2}]] = 0$$

and

$$[e_{2qm} e_p] = [e_{2qm}[e_2 e_1^{p-2}]] = -e_{2qm+p}.$$

In the deflated algebra the first constituent has thus length $2qm$. It follows from the theory of algebras of type 1 that $m = 1$. We will show in Section 5 that algebras of type 2 with $k = q$ come from algebras of type 1.

When $k \equiv \dfrac{1}{2} \pmod{p}$, write

$$k = \frac{qm+1}{2},$$

where p does not divide m. Thus $n = qm + 1$. We want to show that $m = 1$. Suppose otherwise. We have

$$[e_{n-1} e_2] = e_{n+1} \qquad \text{and} \qquad [e_n e_2] = -\frac{1}{2} e_{n+2}.$$

We begin with proving

$$[e_{n+1} e_2] = [e_{n+2} e_2] = \cdots = [e_{n+q-1} e_2] = 0. \tag{10}$$

The identity

$$[e_2 e_1^k e_2] = 0 \tag{11}$$

holds for $k \le n-4$. Note that $n - 4 = qm - 3 \ge 2q - 3$, as $m > 1$.

Let $l < q-1$. Write $l + 1 = \beta p^t$, where $\beta \not\equiv 0 \pmod{p}$. Note that $p^t < q$, so that

$$l + p^t \le q - 2 + p^t \le 2q - 3,$$

and $[e_2 e_1^{l+p^t} e_2] = 0$, by (11).

Suppose first $t > 0$. We compute

$$\begin{aligned}
0 &= [e_{n-1-p^t}[e_2 e_1^{l+p^t} e_2]] \\
&= [e_{n-1-p^t}[e_2 e_1^{l+p^t}] e_2] \\
&\approx \left(\binom{l+p^t}{p^t} + \frac{1}{2} \binom{l+p^t}{p^t+1} \right) [e_{n+l+1} e_2] \\
&= \left(\binom{\beta p^t + p^t - 1}{p^t} + \frac{1}{2} \binom{\beta p^t + p^t - 1}{p^t+1} \right) [e_{n+l+1} e_2] \\
&= \left(\beta + \frac{1}{2}(\beta \cdot (-1)) \right) [e_{n+l+1} e_2] \\
&= \frac{\beta}{2} \cdot [e_{n+l+1} e_2],
\end{aligned}$$

so that $[e_{n+l+1} e_2] = 0$.

Now consider the case when $p^t = 1$, so that $l + 1 \not\equiv 0 \pmod{p}$. An analogous calculation yields

$$0 = \frac{(l+1) \cdot (l+4)}{4} \cdot [e_{n+l+1} e_2].$$

We obtain $[e_{n+l+1} e_2] = 0$, except when $l + 4$ is divisible by p. Note that we may assume $p > 3$ here, since we have already dealt with the case when $l + 1 \equiv 0 \pmod{p}$. We compute

$$\begin{aligned}
0 &= [e_{n-3}[e_2 e_1^{l+2} e_2]] \\
&\approx \left(\binom{l+2}{2} + \frac{1}{2} \cdot \binom{l+2}{3} \right) [e_{n+l+1} e_2] \\
&= [e_{n+l+1} e_2],
\end{aligned}$$

as $p > 3$.

We now reach a contradiction by proving $e_{n+q+1} = e_{qm+q+2} = 0$. Since $n = qm + 1$ is even, m is odd, and $qm + q + 2$ is even. Consider the integer

$$\frac{qm+q+2}{2} = q \cdot \frac{m+1}{2} + 1.$$

Note that

$$q \cdot \frac{m+1}{2} + 1 - 2 = q \cdot \frac{m-1}{2} + q - 1$$

We obtain, using (10),

$$\begin{aligned}
0 &= [e_{q \cdot \frac{m+1}{2}+1} e_{q \cdot \frac{m+1}{2}+1}] = [e_{q \cdot \frac{m+1}{2}+1}[e_2 e_1^{q\frac{m-1}{2}+q-1}]] \\
&= \Bigg((-1)^{q\frac{m-1}{2}-1} \binom{q\frac{m-1}{2}+q-1}{q\frac{m-1}{2}-1} \\
&\qquad + (-1)^{q\frac{m-1}{2}} \binom{q\frac{m-1}{2}+q-1}{q\frac{m-1}{2}} \cdot \left(-\frac{1}{2}\right) \Bigg) \cdot e_{qm+q+2}.
\end{aligned} \tag{12}$$

Now we have

$$\binom{q\frac{m-1}{2}+q-1}{q\frac{m-1}{2}-1} \equiv \binom{q\frac{m-1}{2}+q-1}{q\frac{m-3}{2}+q-1} \equiv \frac{m-1}{2} \pmod{p},$$

while

$$\binom{q\frac{m-1}{2}+q-1}{q\frac{m-1}{2}} = 1.$$

Therefore, up to a sign, the overall coefficient of e_{qm+q+2} in (12) is

$$\frac{m-1}{2}+\frac{1}{2}=\frac{m}{2}\not\equiv 0.$$

This disposes of the case $m>1$, so we obtain

$$k=\frac{q+1}{2}, \quad n=q+1.$$

We will deal with this case in Sections 6 and 8. Remember that when $p=3$ we are taking $q\geq 9$ here. In fact when $q=3$ we get $k=2$, so that $[e_3e_2]\neq 0$, and the algebra does not admit a theory of constituents.

We now deal with the case $k\equiv 1 \pmod{p}$, so $k=1+qm$, where q is a power of p, and $m\not\equiv 0 \pmod{p}$. Thus $n=2qm+2$. We have thus $[e_{n-1}e_2]=e_{n+1}$ and $[e_ne_2]=0$. We want to show that this case does not occur.

Let $1\leq l<q$. Assume by induction

$$[e_ne_2]=[e_{n+1}e_2]=\cdots=[e_{n+l-1}e_2]=0.$$

We compute

$$0=[e_{n-2}[e_2e_1^le_2]]=[e_{n-2}[e_2e_1^l]e_2]=-l[e_{n+l}e_2]. \tag{13}$$

We obtain $[e_{n+l}e_2]=0$ for $l<p$. We can use this and deflation to show that $m=1$. Because of

$$[e_{n-2}e_p]=[e_{n-2}[e_2e_1^{p-2}]]=2e_{n+p-2},$$

the length of the first constituent in the deflated algebra (which is of type 1) is $2qm/p$. If $m>1$, this is not twice a power of p. It follows that $m=1$, and $n=2q+2$.

We now show that $[e_{n+l}e_2]=0$ holds in fact for all $l<q$. Because of the argument of (13), we have to deal with the case $l\equiv 0 \pmod{p}$. If p^t is the highest power of p that divides l, and $l=\beta p^t$, with $\beta\not\equiv 0 \pmod{p}$, we compute

$$\begin{aligned}0&=[e_{n-p^t-1}[e_2e_1^{l+p^t-1}e_2]]\\&=[e_{n-p^t-1}[e_2e_1^{l+p^t-1}]e_2]\\&=\pm\binom{l+p^t-1}{p^t}[e_{n+l}e_2].\end{aligned}$$

Here

$$\binom{l+p^t-1}{p^t}=\binom{\beta p^t+p^t-1}{p^t}\equiv\beta\not\equiv 0 \pmod{p}.$$

We can perform this calculation when $l + r - 1 < 2q - 1$. Note that this holds for $l < q$. We have thus proved

$$[e_n e_2] = [e_{n+1} e_2] = \cdots = [e_{n+q-1} e_2] = 0. \tag{14}$$

Now we use the relation $[e_{n-1} e_2] - e_{n+1} = 0$ to prove $e_{3q+3} = e_{n+q+1} = 0$, a contradiction. We evaluate

$$\begin{aligned} 0 &= [e_q, [e_{n-1} e_2] - e_{n+1}] \\ &= [e_q [e_2 e_1^{2q-1} e_2]] - [e_q [e_2 e_1^{2q+1}]] \end{aligned}$$

Note first that $2q + 1$ is the only value i in the range $2 \le i \le 3q + 1$ for which $[e_i e_2] \ne 0$. Now $[e_q [e_2 e_1^{2q-1} e_2]]$ expands as a combination of commutators of the form $[e_i e_2 e_1^{2q+1-i}]$, for some $q + 2 \le i \le 3q + 1$, so that it vanishes. We obtain

$$\begin{aligned} 0 &= [e_q [e_2 e_1^{2q+1}]] \\ &= (-1)^{q+1} \binom{2q+1}{q+1} e_{3q+3} \\ &= 2 e_{3q+3}. \end{aligned}$$

5. First constituent of length $2q$

This is the case $k = q$ of the previous section. Suppose we have

$$\begin{gathered} [e_i e_2] = 0, \quad \text{for } i < 2q - 1 \\ [e_{2q-1} e_2] = e_{2q+1}, \qquad [e_{2q} e_2] = -e_{2q+2}. \end{gathered}$$

We want to show that the algebra comes from an algebra of type 1 via the procedure described in Section 2, by proving that all constituents have type $(\lambda, -\lambda)$.

Proceeding by induction, assume we have already proved this up to a certain constituent, that ends as

$$[e_m e_2] = \lambda e_{m+2}, \qquad [e_{m+1} e_2] = -\lambda e_{m+3}, \tag{15}$$

for some $\lambda \ne 0$. We first show, also by induction, that $2q$ is an upper bound for the length of the next constituent, and q is a lower bound.

Suppose the next constituent has length greater than $2q$, so that

$$[e_{m+k} e_2] = 0$$

for $2 \le k \le 2q$. We obtain immediately

$$[e_m [e_{2q} e_2]] = [e_m [e_2 e_1^{2q-2} e_2]] = 0,$$

as this is a multiple of $[e_{m+2q} e_2] = 0$. This yields

$$\begin{aligned} 0 &= [e_m, [e_{2q} e_2] + e_{2q+2}] \\ &= [e_m e_{2q+2}] = [e_m [e_2 e_1^{2q}]] = [e_m e_2 e_1^{2q}] \\ &= \lambda e_{m+2+2q}, \end{aligned}$$

a contradiction.

We now prove that the next constituent has length at least q, that is,

$$[e_{m+2} e_2] = [e_{m+3} e_2] = \cdots = [e_{m+q-1} e_2] = 0.$$

This we do more generally for the case when the current constituent is of the general form

$$[e_m e_2] = \mu e_{m+2}, \qquad [e_{m+1} e_2] = \nu e_{m+3}, \tag{16}$$

as this will be useful later in this section. Recall that $\mu \neq 0$ here, but ν might be zero.

If $\nu = 0$ in (16), we compute, proceeding by induction on l, for $0 < l < q - 1$,

$$\begin{aligned}
0 &= [e_{m-1}[e_2 e_1^l e_2]] \\
&= [e_{m-1}[e_2 e_1^l] e_2] \\
&= -l\mu [e_{m+l+1} e_2].
\end{aligned}$$

The coefficient vanishes when $l \equiv 0 \pmod p$. In this case, write $l = \beta p^t$, with $\beta \not\equiv 0 \pmod p$. Note that $p^t < q$ here, so that $l + p^t - 1 < q - 2 + q - 1 < 2q - 3$ and $[e_2 e_1^{l+p^t-1} e_2] = 0$. Also, $[e_{m-p^t} e_2] = \cdots = [e_{m-1} e_2] = 0$, since we are assuming by induction that constituents have length at least q. We compute

$$\begin{aligned}
0 &= [e_{m-p^t}[e_2 e_1^{l+p^t-1} e_2]] \\
&\quad - \binom{l + p^t - 1}{p^t} \mu [e_{m+l+1} e_2].
\end{aligned}$$

Here

$$\binom{l + p^t - 1}{p^t} = \binom{\beta p^t + p^t - 1}{p^t} \equiv \beta \not\equiv 0 \pmod p.$$

Suppose now $\nu \neq 0$. We have first

$$0 = [e_{m-1}[e_2 e_1 e_2]] = -\mu [e_{m+2} e_2],$$

so that $[e_{m+2} e_2] = 0$. We proceed now by induction on l, for $0 < l < q - 2$.

$$\begin{aligned}
0 &= [e_m[e_2 e_1^l e_2]] \\
&= [e_m[e_2 e_1^l] e_2] - [e_m e_2 [e_2 e_1^{l-1}]] \\
&= (\mu - l\nu - \mu(-1)^l)[e_{m+l+2} e_2]. \qquad (17)
\end{aligned}$$

For l even, the coefficient is $-l\nu \neq 0$, so we get $[e_{m+l+2} e_2] = 0$, unless $l \equiv 0 \pmod p$. In this case, we compute

$$\begin{aligned}
0 &= [e_{m+1}[e_2 e_1^{l-1} e_2]] \\
&= [e_{m+1}[e_2 e_1^{l-1}] e_2] - [e_{m+1} e_2 [e_2 e_1^{l-1}]] \\
&= (\nu - (-1)^{l-1}\nu)[e_{m+l+2} e_2].
\end{aligned}$$

As l is even here, the coefficient is $2\nu \neq 0$.

For l odd, the coefficient in (17) is $2\mu - l\nu$. Suppose this vanishes. As $1 \le l < q-2$, we have $q > 3$ here, so that $[e_{m-2} e_2] = 0$. We compute

$$\begin{aligned}
[e_{m-2}[e_2 e_1^{l+2} e_2]] &= \left(\binom{l+2}{2}\mu - \binom{l+2}{3}\nu\right) \cdot [e_{m+l+2} e_2] \\
&= \frac{(l+2)(l+1)}{6}\mu [e_{m+l+2} e_2],
\end{aligned}$$

where we have used the fact that $l\nu = 2\mu$ here. The coefficient vanishes when $l+2 \equiv 0 \pmod{p}$, or $l+1 \equiv 0 \pmod{p}$. (Except possibly when $p = 3$, and $l+1$ or $l+2$ are divisible by 3 but not by 9 – in this case the rest of the discussion is superfluous. Note that $l \not\equiv 0 \pmod{p}$ here, otherwise $\mu = \frac{1}{2} l\nu = 0$. Therefore $l \equiv -1, -2 \pmod{3}$ when $p = 3$, so that $(l+2)(l+1)/6$ is an integer.)

When $l+2 \equiv 0 \pmod{p}$, we have $0 = 2\mu - l\nu = 2(\mu+\nu)$, so that we are in the case of (15), with $\mu = \lambda$ and $\nu = -\lambda$ for some $\lambda \ne 0$. Write $l+2 = \beta p^t$, with $\beta \not\equiv 0$. It is easy to see, with an argument we have employed before, that $l + p^t < 2q - 3$, so that $[e_2 e_1^{l+p^t} e_2] = 0$. We have then

$$\begin{aligned}
0 &= [e_{m-p^t}[e_2 e_1^{l+p^t} e_2]] \\
&= [e_{m-p^t}[e_2 e_1^{l+p^t}] e_2] \\
&= \left(-\binom{l+p^t}{p^t}\lambda + \binom{l+p^t}{p^t+1}(-\lambda)\right) \cdot \cdot [e_{m+l+2} e_2] \\
&= -\lambda \cdot \binom{l+p^t+1}{p^t+1} \cdot [e_{m+l+2} e_2].
\end{aligned}$$

As

$$-\lambda \cdot \binom{l+p^t+1}{p^t+1} = -\lambda \cdot \binom{\beta p^t + p^t - 1}{p^t+1} \equiv \lambda\beta \not\equiv 0 \pmod{p},$$

we get $[e_{m+l+2} e_2] = 0$.

When $l+1 \equiv 0 \pmod{p}$, write $l+1 = \beta p^t$, with $\beta \not\equiv 0 \pmod{p}$. Compute

$$\begin{aligned}
0 &= [e_{m-p^t}[e_2 e_1^{l+p^t} e_2]] \\
&= \left(-\binom{l+p^t}{p^t}\mu + \binom{l+p^t}{p^t+1}\nu\right) \cdot [e_{m+l+1} e_2].
\end{aligned}$$

The coefficient here is, up to a sign, $\beta(\mu+\nu)$. This cannot vanish, otherwise the two relations $\mu + \nu = 0$ and $0 = 2\mu - l\nu = 2\mu + \nu$ would yield $\mu = \nu = 0$, a contradiction.

We now provide the induction step for our assumption that all constituents are of the form $(\lambda, -\lambda)$.

Suppose first the following constituent is of length q. Let

$$[e_{m+q} e_2] = \mu e_{m+q+2}, \qquad \text{and} \qquad [e_{m+q+1} e_2] = \nu e_{m+q+3},$$

We have

$$\begin{aligned}
0 &= [e_{m-1}, [e_{2q-1}e_2] - e_{2q+1}] \\
&= [e_{m-1}, [e_2 e_1^{2q-3} e_2]] - [e_{m-1}, e_{2q+1}] \\
&= [e_{m-1}[e_2 e_1^{2q-3}]e_2] - [e_{m-1}e_2[e_2 e_1^{2q-3}]] - [e_{m-1}[e_2 e_1^{2q-1}]]
\end{aligned}$$

The second term vanishes because $[e_{m-1}e_2] = 0$. The first term is a multiple of $[e_{m+2q-2}e_2] = [e_{(m+q)+q-2}e_2]$. If this is non-zero, it exhibits a constituent of length $q-2$ or $q-1$, whereas we have shown q to be a lower bound for the length of a constituent. Therefore the first term also vanishes.

We are left with

$$\begin{aligned}
0 &= [e_{m-1}[e_2 e_1^{2q-1}]] \\
&\approx (-1)^1 \binom{2q-1}{1} [e_m e_2 e_1^{2q-2}] + (-1)^2 \binom{2q-1}{2} [e_{m+1} e_2 e_1^{2q-3}] \\
&\quad + (-1)^{1+q} \binom{2q-1}{1+q} [e_{m+q} e_2 e_1^{q-2}] + (-1)^{1+q+1} \binom{2q-1}{1+q+1} [e_{m+q+1} e_2 e_1^{q-3}]
\end{aligned}$$

Now the first two binomial coefficients readily evaluate to 1, while for the last two we have, for $l \geq q$,

$$\begin{aligned}
(-1)^{1+l} \binom{2q-1}{1+l} &= (-1)^{l+1} \begin{pmatrix} q & + & q-1 \\ q & + & l-q+1 \end{pmatrix} \\
&\equiv -(-1)^{l-q+1} \binom{q-1}{l-q+1} \\
&= -1.
\end{aligned}$$

We obtain

$$0 = (\lambda - \lambda - \mu - \nu) \cdot e_{m+2q},$$

so that $\nu = -\mu$, as requested.

Suppose now the next constituent has length $l > q$, so that in particular

$$[e_{m+2}e_2] = [e_{m+3}e_2] = \cdots = [e_{m+q}e_2] = 0.$$

We first extend this to show $[e_{m+q+1}e_2] = 0$, so that $l > q+1$. This follows from

$$\begin{aligned}
0 &= [e_m[e_2 e_1^{q-1} e_2]] \\
&= [e_m[e_2 e_1^{q-1}]e_2] - [e_m e_2 [e_2 e_1^{q-1}]] \\
&= (\lambda - \lambda - \lambda) \cdot [e_{m+q+1}e_2].
\end{aligned}$$

Suppose now $[e_{m+l}e_2] = \mu e_{m+l+2}$ and $[e_{m+l+1}e_2] = \nu e_{m+l+3}$. We compute

$$0 = [e_{m+l-q}, [e_{2q-1}e_2] - e_{2q+1}] = [e_{m+l-q}[e_{2q-1}e_2]] - [e_{m+l-q}e_{2q+1}].$$

Keeping in mind that $m + l - q \geq m + 2$, the first term is immediately seen to vanish. We are left with

$$\begin{aligned}
0 &= [e_{m+l-q} e_{2q+1}] \\
&= [e_{m+l-q}[e_2 e_1^{2q-1}]] \\
&= (-1)^q \binom{2q-1}{q} [e_{m+l-q} e_2 e_1^{q-1}] + (-1)^{q+1} \binom{2q-1}{q+1} [e_{m+l-q+1} e_2 e_1^{q-2}] \\
&= (-\mu - \nu) \cdot e_{m+l+1}.
\end{aligned}$$

In this case, too, we obtain $\nu = -\mu$. This completes the induction step.

6. First constituent of length q

Let q be a power of p ($q > 3$), and suppose that $[e_i, e_2] = 0$ for $i = 3, 4, \ldots, q-1$, and that $[e_q, e_2] \neq 0$. By scaling e_2 we may suppose that $[e_q, e_2] = e_{q+2}$. We show that there is a unique infinite dimensional Lie algebra L of type 2 satisfying this condition. The Lie algebra L is defined by the following:

- $[e_i, e_2] = 0$ for $i = 3, 4, \ldots, q-1$,
- $[e_q, e_2] = e_{q+2}$, $[e_{q+1}, e_2] = -\frac{1}{2} e_{q+3}$,
- $[e_{kq}, e_2] = \frac{1}{2} e_{kq+2}$, $[e_{kq+1}, e_2] = -\frac{1}{2} e_{kq+3}$ for $k = 2, 3, \ldots$,
- $[e_k, e_2] = 0$ for $k > q + 1$ unless $k = 0 \pmod{q}$ or $k = 1 \pmod{q}$.

Note that in this Lie algebra, if $m > q$ and $n \geq 1$ then

$$[[e_m, e_n, e_1^q] = [e_m, e_1^q, e_n]$$

so that

$$[[e_m, e_{n+q}] = [e_m, [e_n, e_1^q]] = 0$$

It follows that if $m, n > q$ then $[e_m, e_n] = 0$, so that the Lie algebra is soluble. We give a construction of L in Section 8, and we make use of the existence of L in the following way. In L we have $[e_n, e_2] = \mu_n e_{n+2}$ for $n > 2$, where $\mu_n = 0, 1, \frac{1}{2}$, or $-\frac{1}{2}$ as described above. Suppose that we have a Lie algebra M of type 2, where M is spanned by $\{e_i \,|\, i \geq 1\}$, with $[e_i, e_1] = e_{i+1}$ for $i > 1$ and $[e_n, e_2] = \mu_n e_{n+2}$ for $2 < n < 2m - 2$. Then the relation $[e_m, e_m] = 0$ gives

$$\begin{aligned}
0 &= [e_m, [e_2, e_1^{m-2}]] \\
&= \sum_{k=0}^{m-2} (-1)^k \binom{m-2}{k} [e_m, e_1^k, e_2, e_1^{m-2-k}] \\
&= \sum_{k=0}^{m-3} (-1)^k \binom{m-2}{k} \mu_{m+k} e_{2m} + (-1)^{m-2} [e_{2m-2}, e_2].
\end{aligned}$$

So $[e_{2m-2}, e_2] = \mu e_{2m}$ for some μ which is uniquely determined by $\{\mu_k \,|\, m \leq k < 2m - 2\}$. The existence of L implies that $\mu = \mu_{2m-2}$.

So we assume that $[e_i, e_2] = 0$ for $i = 3, 4, \ldots, q-1$, and that $[e_q, e_2] = e_{q+2}$. The argument just given implies that

$$[e_{q+1}, e_2] = \mu_{q+1} e_{q+3} = -\frac{1}{2} e_{q+3}.$$

Since $q > 3$, $[e_1, e_2, e_2] = 0$, and so

$$0 = [e_{q-1}, [e_1, e_2, e_2]] = [e_q, e_2, e_2] = [e_{q+2}, e_2].$$

It follows that $[e_{q+3}, e_2] = \mu_{q+3} e_{q+5} = 0$.

We now show by induction that $[e_k, e_2] = 0$ for $k = q+2, q+3, \dots, 2q-1$. We have established the cases $k = q+2$ and $q+3$. So suppose that $q+3 < m < 2q$, and suppose that $[e_k, e_2] = 0$ for $k = q+2, q+3, \dots, m-1$.

Using the argument above, it is only necessary to consider the case when m is odd. Then

$$\begin{aligned}
0 &= [e_{q+1}, [e_2, e_1^{m-q-3}, e_2]] \\
&= [e_{q+1}, [e_2, e_1^{m-q-3}], e_2] - [e_{q+1}, e_2, [e_2, e_1^{m-q-3}]] \\
&= [e_{q+1}, e_2, e_1^{m-q-3}, e_2] - (-1)^{m-q-3} [e_{q+1}, e_2, e_1^{m-q-3}, e_2] \\
&= -[e_m, e_2].
\end{aligned}$$

So $[e_k, e_2] = 0$ for $k = q+2, q+3, \dots, 2q-1$, as claimed. Also

$$[e_{2q}, e_2] = \mu_{2q} e_{2q+2} = \frac{1}{2} e_{2q+2}.$$

The equations obtained so far leave $[e_{2q+1}, e_2]$ undetermined, and so we suppose that

$$[e_{2q+1}, e_2] = \lambda e_{2q+3},$$

for some λ. We will show below that λ must equal $-\frac{1}{2}$ or $-\frac{1}{4}$, but first we show that $[e_k, e_2] = 0$ for $2q+1 < k < 3q$. It is convenient to subdivide the proof of this into the case when $\lambda = 0$ and the case when $\lambda \neq 0$.

First consider the case when $\lambda = 0$.

The equation $[[e_2, e_1^q], [e_2, e_1^q]] = 0$ gives

$$[e_2, e_1^q, e_2, e_1^q] = [e_2, e_1^{2q}, e_2],$$

which implies that $[e_{2q+2}, e_2] = 0$. And

$$[e_{2q}, [e_1, e_2, e_2]] = 0$$

gives

$$-[e_{2q+3}, e_2] = 0,$$

So we assume that $2q+3 < m < 3q$, and that $[e_k, e_2] = 0$ for $2q < k < m$. If m is odd then

$$0 = [e_{2q}, [e_2, e_1^{m-2q-2}, e_2]] = [e_m, e_2].$$

If m is even and $m < 3q-1$ then $[e_{2q-1}, [e_2, e_1^{m-2q-1}, e_2]] = 0$ gives

$$(m - 2q - 1)[e_m, e_2] = 0. \tag{18}$$

Also if $2q+3<m<3q$ then $[e_{m-q},e_2]=0$, and so the equation

$$[e_{m-q},[e_2,e_1^{q-2},e_2]]=[e_{m-q},[e_2,e_1^q]]$$

gives

$$((3q-m+1)\frac{1}{2}+1)[e_m,e_2]=0. \tag{19}$$

From (18) we see that if m is even and $2q+3<m<3q-1$ then $[e_m,e_2]=0$ unless $m=1 \pmod p$. But (19) shows that $[e_m,e_2]=0$ in the case when $m=1 \pmod p$, as well as in the case when $m=3q-1$. So $[e_k,e_2]=0$ for $2q+1<k<3q$ in the case when $\lambda=0$.

So suppose that $\lambda\neq 0$. As above, we want to show that $[e_k,e_2]=0$ for $2q+1<k<3q$. Since we need the following argument several times, it is convenient to put it in the form of a lemma.

Lemma. *Let $t\geq 1$ and let $q=p^s>3$. Suppose that $[e_k,e_2]=0$ for $1<k<2tq$ unless $k=0 \pmod q$ or $k=1 \pmod q$, that $[e_{2tq},e_2]=\alpha e_{2tq+2}$ for some $\alpha\neq 0$, and that $[e_{2tq+1},e_2]=\lambda e_{2tq+3}$ for some $\lambda\neq 0$. Then $[e_{2tq+k},e_2]=0$ for $1<k<q$.*

Proof. The case $k=2$ follows from

$$0=[e_{2tq-1},[e_1,e_2,e_2]]=\alpha[e_{2tq+2},e_2].$$

Now suppose by induction that m is odd, that $3\leq m\leq q-2$, and that $[e_{2tq+k},e_2]=0$ for all k such that $1<k<m$. We show that $[e_{2tq+m},e_2]=[e_{2tq+m+1},e_2]=0$, and this establishes the lemma by induction on (odd) m.

First we have

$$0=[e_{2tq+1},[e_2,e_1^{m-2},e_2]]=2\lambda[e_{2tq+m+1},e_2]-\lambda(m-2)[e_{2tq+m},e_2,e_1]. \tag{20}$$

If we let $d=\frac{m+3}{2}$, we also have

$$\begin{aligned} 0 &= [e_{tq+d},e_{tq+d}] \\ &= [e_{tq+d},[e_d,e_1^{tq}]] \\ &= \sum_{r=0}^{t}(-1)^r\binom{t}{r}[e_{(t+r)q+d},e_d,e_1^{(t-r)q}]. \end{aligned}$$

Now our hypotheses imply that $[e_{(t+r)q+d},e_d]=0$ if $r<t$. So this equation implies that $[e_{2tq+d},e_d]=0$ also. Since $e_d=[e_2,e_1^{d-2}]$ this gives

$$\sum_{r=0}^{d-2}(-1)^r\binom{d-2}{r}[e_{2tq+d+r},e_2,e_1^{d-2-r}]=0.$$

But our inductive hypothesis implies that $[e_{2tq+d+r},e_2]=0$ for $r<d-3$. So we obtain

$$[e_{2tq+m+1},e_2]=(d-2)[e_{2tq+m},e_2,e_1]. \tag{21}$$

Since $d-2=\frac{m-1}{2}$ (20) and (21) imply that $[e_{2tq+m+1},e_2]=[e_{2tq+m},e_2,e_1]=0$, which also implies that $[e_{2tq+m},e_2]=0$. This completes the proof of the lemma. □

So $[e_k, e_2] = 0$ for $2q+1 < k < 3q$, whatever the value of λ.
Now consider the equation

$$[e_{2q}, [e_2, e_1^{q-2}, e_2]] = [e_{2q}, [e_2, e_1^q]].$$

This gives

$$\begin{aligned} 0 &= [e_{2q}, [e_2, e_1^{q-2}], e_2] - [e_{2q}, e_2, [e_2, e_1^{q-2}]] - [e_{2q}, e_2, e_1^q] + [e_{2q}, e_1^q, e_2] \\ &= \frac{1}{2}[e_{3q}, e_2] + 2\lambda[e_{3q}, e_2] + \frac{1}{2}[e_{3q}, e_2] - \frac{1}{2}e_{3q+2} + [e_{3q}, e_2] \\ &= (2+2\lambda)[e_{3q}, e_2] - \frac{1}{2}e_{3q+2}, \end{aligned}$$

which implies that

$$e_{3q+3} = (4+4\lambda)[e_{3q}, e_2, e_1] \tag{22}$$

And

$$[e_{2q+1}, [e_2, e_1^{q-2}, e_2]] = [e_{2q+1}, [e_2, e_1^q]]$$

gives

$$2\lambda[e_{3q+1}, e_2] - \lambda(q-2)[e_{3q}, e_2, e_1] = \lambda e_{3q+3} - [e_{3q+1}, e_2]. \tag{23}$$

In addition, the equation

$$[[e_2, e_1^{\frac{3q-1}{2}}], [e_2, e_1^{\frac{3q-1}{2}}]] = 0$$

gives

$$\begin{aligned} 0 = {} & \left((-1)^{\frac{q-3}{2}}\binom{\frac{3q-1}{2}}{\frac{q-3}{2}}\frac{1}{2} + (-1)^{\frac{q-1}{2}}\binom{\frac{3q-1}{2}}{\frac{q-1}{2}}\lambda\right)e_{3q+3} \\ & + (-1)^{\frac{3q-3}{2}}\frac{3q-1}{2}[e_{3q}, e_2, e_1] + (-1)^{\frac{3q-1}{2}}[e_{3q+1}, e_2] \end{aligned} \tag{24}$$

Now

$$\binom{\frac{3q-1}{2}}{\frac{q-3}{2}} = \binom{q + \frac{q-1}{2}}{\frac{q-1}{2} - 1} = \frac{q-1}{2} \pmod{p},$$

and

$$\binom{\frac{3q-1}{2}}{\frac{q-1}{2}} = \binom{q + \frac{q-1}{2}}{\frac{q-1}{2}} = 1 \bmod p,$$

So (24) gives

$$(-1)^{\frac{q-1}{2}}\left(\frac{1}{4} + \lambda\right)e_{3q+3} - (-1)^{\frac{3q-3}{2}}\frac{1}{2}[e_{3q}, e_2, e_1] + (-1)^{\frac{3q-1}{2}}[e_{3q+1}, e_2] = 0,$$

which implies that

$$\left(\frac{1}{4} + \lambda\right)e_{3q+3} - \frac{1}{2}[e_{3q}, e_2, e_1] - [e_{3q+1}, e_2] = 0. \tag{25}$$

From (22) and (25) we obtain

$$[e_{3q+1}, e_2] = ((4+4\lambda)(\frac{1}{4}+\lambda) - \frac{1}{2})[e_{3q}, e_2, e_1].$$

So (23) gives

$$(2\lambda+1)((4+4\lambda)(\frac{1}{4}+\lambda) - \frac{1}{2}) + 2\lambda = \lambda(4+4\lambda),$$

which implies that $\lambda = -\frac{1}{2}$ or $-\frac{1}{4}$.

If $\lambda = -\frac{1}{2}$ then $[e_{3q}, e_2] = \frac{1}{2}e_{3q+2}$, and $[e_{3q+1}, e_2] = -\frac{1}{2}e_{3q+3}$. If $\lambda = -\frac{1}{4}$ and $p = 3$, then (22) gives $e_{3q+3} = 0$, so $\lambda = -\frac{1}{2}$ is the only possibility when $p = 3$. If $\lambda = -\frac{1}{4}$ and $p \neq 3$, then we have $[e_{3q}, e_2] = \frac{1}{3}e_{3q+2}$ and $[e_{3q+1}, e_2] = -\frac{1}{6}e_{3q+3}$.

Thus we have established that if q is a power of p ($q > 3$), and $[e_i, e_2] = 0$ for $i = 3, 4, \ldots, q-1$, and $[e_q, e_2] = e_{q+2}$, then

$$[e_{q+1}, e_2] = -\frac{1}{2}e_{q+3},$$

$$[e_k, e_2] = 0 \text{ for } q+1 < k < 2q,$$

$$[e_{2q}, e_2] = \frac{1}{2}e_{2q+2},$$

$$[e_{2q+1}, e_2] = \lambda e_{2q+3} \text{ where } \lambda = -\frac{1}{2} \text{ or } \lambda = -\frac{1}{4},$$

$$[e_k, e_2] = 0 \text{ for } 2q+1 < k < 3q,$$

$$[e_{3q}, e_2] = \begin{cases} \frac{1}{2}e_{3q+2} \text{ when } \lambda = -\frac{1}{2} \\ \frac{1}{3}e_{3q+2} \text{ when } \lambda = -\frac{1}{4} \end{cases},$$

$$[e_{3q+1}, e_2] = \begin{cases} -\frac{1}{2}e_{3q+3} \text{ when } \lambda = -\frac{1}{2} \\ -\frac{1}{6}e_{3q+3} \text{ when } \lambda = -\frac{1}{4} \end{cases}.$$

Furthermore, the case $\lambda = -\frac{1}{4}$ can only arise when $p \neq 3$.

6.1. **Generic step for $\lambda = -\frac{1}{2}$.** We assume that q is a power of p ($q > 3$) and we assume that

- $[e_i, e_2] = 0$ for $i = 3, 4, \ldots, q-1$,
- $[e_q, e_2] = e_{q+2}$, $[e_{q+1}, e_2] = -\frac{1}{2}e_{q+3}$,
- $[e_{kq}, e_2] = \frac{1}{2}e_{kq+2}$, $[e_{kq+1}, e_2] = -\frac{1}{2}e_{kq+3}$ for $k = 2, 3, \ldots, 2n-1$ ($n \geq 2$),
- $[e_k, e_2] = 0$ for $q+1 < k < (2n-1)q$ unless $k = 0 \pmod q$ or $k = 1 \pmod q$.

We show that

- $[e_{(2n-1)q+k}, e_2] = 0$ for $1 < k < q$,
- $[e_{2nq}, e_2] = \frac{1}{2}e_{2nq+2}$, $[e_{2nq+1}, e_2] = \lambda e_{2nq+3}$ where $\lambda = -\frac{1}{2}$ or $-\frac{1}{4}$,
- $[e_{2nq+k}, e_2] = 0$ for $1 < k < q$,
- if $\lambda = -\frac{1}{2}$ then $[e_{(2n+1)q}, e_2] = \frac{1}{2}e_{(2n+1)q+2}$ and $[e_{(2n+1)q+1}, e_2] = -\frac{1}{2}e_{(2n+1)q+3}$,
- if $\lambda = -\frac{1}{4}$ then $[e_{(2n+1)q}, e_2] = \frac{1}{3}e_{(2n+1)q+2}$ and $[e_{(2n+1)q+1}, e_2] = -\frac{1}{6}e_{(2n+1)q+3}$.

First we show that $[e_{(2n-1)q+k}, e_2] = 0$ for $1 < k < q$. Since $[e_{(2n-2)q+k}, e_2] = 0$, the equation

$$[e_{(2n-2)q+k}, [e_2, e_1^{q-2}, e_2]] = [e_{(2n-2)q+k}, [e_2, e_1^q]]$$

gives

$$(-1)^{q-k}\left(\binom{q-2}{q-k} + \binom{q-2}{q-k+1}\right)\frac{1}{2}[e_{(2n-1)q+k}, e_2] = -[e_{(2n-1)q+k}, e_2].$$

This implies that $[e_{(2n-1)q+k}, e_2] = 0$, since

$$\begin{aligned} 1 + (-1)^{q-k}\binom{q-2}{q-k}\frac{1}{2} + (-1)^{q-k}\binom{q-2}{q-k+1}\frac{1}{2} &= \\ = 1 + (q-k+1)\frac{1}{2} - (q-k+2)\frac{1}{2} \pmod{p} &= \\ &= \frac{1}{2} \pmod{p}. \end{aligned}$$

Next consider the equation

$$[e_{(2n-1)q}, [e_2, e_1^{q-2}, e_2]] = [e_{(2n-1)q}, [e_2, e_1^q]]$$

This gives

$$\begin{aligned} & 2[[e_{(2n-1)q}, e_2, e_1^{q-2}, e_2] + 2[[e_{(2n-1)q+1}, e_2, e_1^{q-3}, e_2] \\ = & [e_{(2n-1)q}, e_2, e_1^q] - [e_{(2n-1)q}, e_1^q, e_2] \end{aligned}$$

which implies that

$$[e_{2nq}, e_2] = \frac{1}{2}e_{2nq+2}.$$

The equations obtained so far leave $[e_{2nq+1}, e_2]$ undetermined, and so we suppose that

$$[e_{2nq+1}, e_2] = \lambda e_{2nq+3},$$

for some λ. We will show below that λ must equal $-\frac{1}{2}$ or $-\frac{1}{4}$.

First note that the lemma implies that if $\lambda \neq 0$ then $[e_{2nq+k}, e_2] = 0$ for $1 < k < q$. We show that $[e_{2nq+k}, e_2] = 0$ for $1 < k < q$ in the case $\lambda = 0$ also. So suppose that $\lambda = 0$.

$$0 = [e_{2nq-1}, [e_1, e_2, e_2]] = \frac{1}{2}[e_{2nq+2}, e_2].$$

Also

$$\begin{aligned} 0 &= [e_{2nq}, [e_1, e_2, e_2]] \\ &= [e_{2nq}, e_1, e_2, e_2] - 2[e_{2nq}, e_2, e_1, e_2]] + [e_{2nq}, e_2, e_2, e_1] \\ &= -[e_{2nq+3}, e_2], \end{aligned}$$

so $[e_{2nq+3}, e_2] = 0$.

We assume that $3 < m < q$, and that $[e_{2nq+k}, e_2] = 0$ for $1 < k < m$. If m is odd then

$$0 = [e_{2nq}, [e_2, e_1^{m-2}, e_2]] = [e_{2nq+m}, e_2].$$

If m is even and $m < q-1$ then $[e_{2nq-1}, [e_2, e_1^{m-1}, e_2]] = 0$ gives

$$(m-1)[e_{2nq+m}, e_2] = 0. \tag{26}$$

Also if $3 < m < q$ then $[e_{(2n-1)q+m}, e_2] = 0$, and so the equation

$$[e_{(2n-1)q+m}, [e_2, e_1^{q-2}, e_2]] = [e_{(2n-1)q+m}, [e_2, e_1^q]]$$

gives

$$((q-m+1)\frac{1}{2} + 1)[e_{2nq+m}, e_2] = 0. \tag{27}$$

From (26) we see that if m is even and $3 < m < q-1$ then $[e_{2nq+m}, e_2] = 0$ unless $m = 1 \pmod p$. But (27) shows that $[e_{2nq+m}, e_2] = 0$ in the case when $m = 1 \pmod p$, as well as in the case when $m = q-1$. So $[e_{2nq+k}, e_2] = 0$ for $1 < k < q$ in the case when $\lambda = 0$, as well as in the case $\lambda \neq 0$.

Now consider the equation

$$[e_{2nq}, [e_2, e_1^{q-2}, e_2]] = [e_{2nq}, [e_2, e_1^q]].$$

This gives

$$e_{(2n+1)q+3} = (4+4\lambda)[e_{(2n+1)q}, e_2, e_1] \tag{28}$$

in exactly the same way as (22) was obtained from $[e_{2q}, [e_2, e_1^{q-2}, e_2]] = [e_{2q}, [e_2, e_1^q]]$. And

$$[e_{2nq+1}, [e_2, e_1^{q-2}, e_2]] = [e_{2nq+1}, [e_2, e_1^q]]$$

gives

$$2\lambda[e_{(2n+1)q+1}, e_2] - \lambda(q-2)[e_{(2n+1)q}, e_2, e_1] = \lambda e_{(2n+1)q+3} - [e_{(2n+1)q+1}, e_2]. \tag{29}$$

Now consider the equation

$$[[e_{nq+2}, e_1^{\frac{q-1}{2}}], [e_{nq+2}, e_1^{\frac{q-1}{2}}]] = 0 \tag{30}$$

If we expand $[[e_{nq+2}, e_1^{\frac{q-1}{2}}], [e_{nq+2}, e_1^{\frac{q-1}{2}}]]$ we obtain a sum of the form

$$\sum_{r=\frac{q-1}{2}}^{q-1} \alpha_r [e_{nq+2+r}, e_{nq+2}, e_1^{q-1-r}].$$

Now

$$\begin{aligned} & [e_{nq+2+r}, e_{nq+2}] \\ = \; & [e_{nq+2+r}, [e_2, e_1^{nq}]] \\ = \; & \sum_{s=0}^{n} (-1)^s \binom{n}{s} [e_{nq+2+r}, e_1^{sq}, e_2, e_1^{(n-s)q}]. \end{aligned}$$

If $r < q-2$, then $[e_{nq+2+r}, e_1^{sq}, e_2, e_1^{(n-s)q}] = 0$ for all s. If $r = q-2$ then

$$[e_{nq+2+r}, e_1^{sq}, e_2, e_1^{(n-s)q}] = \frac{1}{2} e_{(2n+1)q+2}$$

for $s < n$, and

$$[e_{nq+2+r}, e_1^{sq}, e_2, e_1^{(n-s)q}] = [e_{(2n+1)q}, e_2]$$

for $s = n$. It follows that if $r = q - 2$ then

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s} [e_{nq+2+r}, e_1^{sq}, e_2, e_1^{(n-s)q}] = (-1)^n (\frac{1}{2} e_{(2n+1)q+2} - [e_{(2n+1)q}, e_2]).$$

Similarly, if $r = q - 1$ then

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s} [e_{nq+2+r}, e_1^{sq}, e_2, e_1^{(n-s)q}] = (-1)^n (\lambda e_{(2n+1)q+3} - [e_{(2n+1)q+1}, e_2]).$$

So (30) gives

$$\frac{1}{2}(\frac{1}{2} e_{(2n+1)q+3} - [e_{(2n+1)q}, e_2, e_1]) + \lambda e_{(2n+1)q+3} - [e_{(2n+1)q+1}, e_2] = 0,$$

which implies that

$$\left(\frac{1}{4} + \lambda\right) e_{(2n+1)q+3} - \frac{1}{2}[e_{(2n+1)q}, e_2, e_1] - [e_{(2n+1)q+1}, e_2] = 0. \tag{31}$$

Equations (28), (29) and (31) imply that $\lambda = -\frac{1}{2}$ or $-\frac{1}{4}$ in exactly the same way as equations (22), (23) and (25) do. They similarly imply that if $\lambda = -\frac{1}{2}$ then $[e_{(2n+1)q}, e_2] = \frac{1}{2} e_{(2n+1)q+2}$, and $[e_{(2n+1)q+1}, e_2] = -\frac{1}{2} e_{(2n+1)q+3}$. If $\lambda = -\frac{1}{4}$ and $p = 3$, then (28) gives $e_{(2n+1)q+3} = 0$, so $\lambda = -\frac{1}{2}$ is the only possibility when $p = 3$. If $\lambda = -\frac{1}{4}$ and $p \neq 3$, then we have $[e_{(2n+1)q}, e_2] = \frac{1}{3} e_{(2n+1)q+2}$ and $[e_{(2n+1)q+1}, e_2] = -\frac{1}{6} e_{(2n+1)q+3}$.

This establishes the generic step for $\lambda = -\frac{1}{2}$.

6.2. **Generic step for $\lambda = -\frac{1}{4}$.** We assume that q is a power of p ($p > 3$) and we assume that

- $[e_i, e_2] = 0$ for $i = 3, 4, \dots, q - 1$,
- $[e_q, e_2] = e_{q+2}$, $[e_{q+1}, e_2] = -\frac{1}{2} e_{q+3}$,
- $[e_{kq}, e_2] = \frac{1}{2} e_{kq+2}$, $[e_{kq+1}, e_2] = -\frac{1}{2} e_{kq+3}$ for $k = 2, 3, \dots, 2n - 1$ ($n \geq 2$),
- $[e_{2nq}, e_2] = \frac{1}{2} e_{2nq+2}$, $[e_{2nq+1}, e_2] = -\frac{1}{4} e_{2nq+3}$,
- There exists s with $1 \leq s < p-2$ such that $[e_k, e_2] = 0$ for $q+1 < k < (2n+s)q$ unless $k = 0 \pmod q$ or $k = 1 \pmod q$,
- $[e_{(2n+k)q}, e_2] = \frac{1}{k+2} e_{(2n+k)q+2}$ and $[e_{(2n+k)q+1}, e_2] = -\frac{1}{2(k+2)} e_{(2n+k)q+3}$ for $k = 1, 2, \dots, s$.

Note that this situation arises from the case $\lambda = -\frac{1}{4}$ of the last section, with $s = 1$.

We show that $[e_{(2n+s)q+k}, e_2] = 0$ for $1 < k < q$. In addition we show that if $s < p - 3$ then $[e_{(2n+s+1)q}, e_2] = \frac{1}{s+3} e_{(2n+s+1)q+2}$, $[e_{(2n+s+1)q+1}, e_2] = -\frac{1}{2(s+3)} e_{(2n+s+1)q+3}$, and we show that if $s = p - 3$ then $e_{(2n+s+1)q+2} = 0$. This contradiction shows that the case $\lambda = -\frac{1}{4}$ cannot arise in an infinite dimensional Lie algebra of type 2.

For the moment we suppose that $s < p - 3$.

First we show that $[e_{(2n+s)q+k}, e_2] = 0$ for $1 < k < q$. The case $k = 2$ follows from

$$0 = [e_{(2n+s)q-1}, [e_1, e_2, e_2]] = \frac{1}{s+2}[e_{(2n+s)q+2}, e_2].$$

For $k = 3$ we have

$$[e_{(2n+s-1)q+3}, [e_2, e_1^{q-2}, e_2]] = [e_{(2n+s-1)q+3}, [e_2, e_1^q]]$$

which gives

$$((q-2)\frac{1}{s+2} - (q-1)\frac{1}{2(s+2)})[e_{(2n+s)q+3}, e_2] = -[e_{(2n+s)q+3}, e_2],$$

and so implies that $[e_{(2n+s)q+3}, e_2] = 0$ unless $2s + 1 = 0 \pmod p$. We also have

$$\begin{aligned} 0 &= [e_{(2n+s)q}, [e_1, e_2, e_2]] \\ &= (-\frac{1}{2(s+2)} - \frac{2}{s+2})[e_{(2n+s)q+3}, e_2], \end{aligned}$$

which implies that $[e_{(2n+s)q+3}, e_2] = 0$ unless $p = 5$. Now if $p = 5$ then our assumption that $1 \le s < p - 3$ implies that $s = 1$ so that $2s + 1 \ne 0$ mod 5. So $[e_{(2n+s)q+3}, e_2] = 0$ in every case.

Now suppose that $3 < m < q$ and that $[e_{(2n+s)q+k}, e_2] = 0$ for all k such that $1 < k < m$. If m is even then

$$0 = [e_{(2n+s)q+1}, [e_2, e_1^{m-3}, e_2]] = -\frac{1}{s+2}[e_{(2n+s)q+m}, e_2].$$

So we may assume that m is odd. In this case we have

$$\begin{aligned} 0 &= [e_{(2n+s)q}, [e_2, e_1^{m-2}, e_2]] \\ &= (\frac{2}{s+2} - (m-2)\frac{-1}{2(s+2)})[e_{(2n+s)q+m}, e_2], \end{aligned}$$

so $[e_{(2n+s)q+m}, e_2] = 0$ unless $m = -2 \pmod p$. We also have

$$[e_{(2n+s-1)q+m}, [e_2, e_1^{q-2}, e_2]] = [e_{(2n+s-1)q+m}, [e_2, e_1^q]].$$

This implies that

$$((q-m+1)\frac{1}{s+2} - (q-m+2)\frac{1}{2(s+2)} + 1)[e_{(2n+1)q+m}, e_2] = 0,$$

which implies that $[e_{(2n+s)q+m}, e_2] = 0$ unless $m = 2(s+2) \pmod p$. Since $s < p - 3$, $[e_{(2n+s)q+m}, e_2] = 0$ in every case.

Next consider the equation

$$[e_{(2n+s)q}, [e_2, e_1^{q-2}, e_2]] = [e_{(2n+s)q}, [e_2, e_1^q]].$$

This implies that

$$(\frac{2}{s+2} - \frac{2}{2(s+2)} + 1)[e_{(2n+s+1)q}, e_2] = \frac{1}{s+2}e_{(2n+s+1)q+2},$$

and hence that

$$[e_{(2n+s+1)q}, e_2] = \frac{1}{s+3}e_{(2n+s+1)q+2}.$$

And the equation

$$[e_{(2n+s)q+1}, [e_2, e_1^{q-2}, e_2]] = [e_{(2n+s)q+1}, [e_2, e_1^q]]$$

implies that

$$\begin{aligned} &(-\frac{2}{2(s+2)} + 1)[e_{(2n+s+1)q+1}, e_2] + \frac{1}{2(s+2)}(q-2)[e_{(2n+s+1)q}, e_2, e_1] \\ = \quad &-\frac{1}{2(s+2)} e_{(2n+s+1)q+3}. \end{aligned}$$

So

$$[e_{(2n+s+1)q+1}, e_2] = -\frac{1}{2(s+3)} e_{(2n+s+1)q+3}.$$

Finally we consider the case when $s = p-3$. Let $2t = 2n+s = 2n+p-3$. Then $[e_{2tq}, e_2] = -e_{2tq+2}$ and $[e_{2tq+1}, e_2] = \frac{1}{2}e_{2tq+3}$.

The lemma implies that $[e_{2tq+k}, e_2] = 0$ for $1 < k < q$.

Consider the equation

$$[e_{2tq}, [e_2, e_1^{q-2}, e_2]] = [e_{2tq}, [e_2, e_1^q]].$$

This implies that

$$(-2 - \frac{q-2}{2} + 1)[e_{(2t+1)q}, e_2] = -e_{(2t+1)q+2},$$

and hence that

$$e_{(2t+1)q+2} = 0.$$

7. The case $[e_3 e_2] \neq 0$ and $p = 3$

Let L be an $\mathbb{N}$-graded Lie algebra of maximal class over a field $\mathbf{F}$ of characteristic 3, where L has basis $\{e_i \mid i = 1, 2, \dots\}$, with $[e_i, e_1] = e_{i+1}$ for $i > 1$. We consider the case when $[e_3, e_2] \neq 0$. By rescaling e_2 we may assume that $[e_3, e_2] = e_5$, which implies that $[e_4, e_2] = e_6$ but leaves $[e_5, e_2]$ undetermined. We show that for every $\lambda \in \mathbf{F}$ there is a unique infinite dimensional soluble algebra $L(\lambda)$ of type 2 satisfying these relations, together with the relation $[e_5, e_2] = \lambda e_7$. The algebra $L(\lambda)$ has basis $\{e_i \mid i = 1, 2, \dots\}$, and satisfies the following relations:

$$\begin{cases} [e_i, e_1] = e_{i+1}, & \text{for } i > 1, \\ [e_3, e_2] = e_5, & \\ [e_{3k+1}, e_2] = e_{3k+3}, [e_{3k+2}, e_2] = \lambda e_{3k+4}, & \\ \qquad [e_{3k+3}, e_2] = (-1-\lambda)e_{3k+5}, & \text{for } k \geq 1, \\ [e_k, e_3] = (1-\lambda)e_{k+3}, & \text{for } k \geq 4, \\ [e_k, e_m] = 0, & \text{for } k, m \geq 4. \end{cases} \tag{32}$$

We give a construction of $L(\lambda)$ in Section 9, but in fact it is easy to show directly that these relations (together with the relations $[e_i, e_i] = 0$, $[e_i, e_j] + [e_j, e_i] = 0$) imply the Jacobi relations

$$[e_i, e_j, e_k] + [e_j, e_k, e_i] + [e_k, e_i, e_j] = 0.$$

Note that $L(1) \cong m_2$ and that $L(-1)$ is the analogue for $q = 3$ of the algebra constructed in Section 6.

So we suppose that L has basis $\{e_i \,|\, i = 1, 2, \dots\}$, with $[e_i, e_1] = e_{i+1}$ for $i > 1$, $[e_3, e_2] = e_5$, $[e_4, e_2] = e_6$, $[e_5, e_2] = \lambda e_7$. We show that if $n \geq 4$ then $[e_n, e_2] = \mu_n e_{n+2}$, where

$$\mu_n = \begin{cases} 1 \text{ if } n = 1 \pmod 3 \\ \lambda \text{ if } n = 2 \pmod 3 \\ -1-\lambda \text{ if } n = 0 \pmod 3). \end{cases}$$

The fact that $[e_k, e_3] = (1-\lambda)e_{k+3}$ for $k \geq 4$, and that $[e_k, e_m] = 0$ for $k, m \geq 4$, follows easily from this.

We will make use of the following argument. Suppose that we have shown that $[e_n, e_2] = \mu_n e_{n+2}$ for all n with $4 \leq n < 2m$. Then the relation $[e_{m+1}, e_{m+1}] = 0$ implies that

$$\begin{aligned} 0 &= [e_{m+1}, [e_2, e_1^{m-1}]] \\ &= \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} [e_{m+1}, e_1^k, e_2, e_1^{m-1-k}], \end{aligned}$$

and so $[e_{2m}, e_2]$ is determined by the values of $[e_n, e_2]$ for $m+1 \leq n < 2m$. So $[e_{2m}, e_2] = \mu e_{2m+2}$, for some μ which is uniquely determined by $\{\mu_n \,|\, m+1 \leq n < 2m\}$. But $L(\lambda)$ is a Lie algebra which satisfies $[e_n, e_2] = \mu_n e_{n+2}$ for all $n \geq 4$. So $\mu = \mu_{2m}$. In particular, this argument implies that $[e_6, e_2] = (-1-\lambda)e_8$.

Now suppose that $[e_n, e_2] = \mu_n e_{n+2}$ for $4 \leq n < m$ for some $m \geq 7$. We show that this implies that $[e_m, e_2] = \mu_m e_{m+2}$. By the argument above, we only need to consider the case when m is odd. We use the fact that $[e_2, e_1^3] + [e_1, e_2, e_2] = 0$. So

$$\begin{aligned} 0 &= [e_{m-3}, [e_2, e_1^3]] + [e_{m-3}, [e_1, e_2, e_2]] \\ &= [e_{m-3}, e_2, e_1^3] - [e_{m-3}, e_1^3, e_2] + \\ &\quad + [e_{m-3}, e_1, e_2, e_2] + [e_{m-3}, e_2, e_1, e_2] + [e_{m-3}, e_2, e_2, e_1] \\ &= \mu_{m-3}(1 + \mu_{m-1})e_{m+2} - (1 - \mu_{m-2} - \mu_{m-3})[e_m, e_2] \\ &= \mu_{m-3}(1 + \mu_{m-1})e_{m+2} - (1 + \mu_{m-1})[e_m, e_2]. \end{aligned}$$

Provided $1 + \mu_{m-1} \neq 0$, this gives $[e_m, e_2] = \mu_{m-3}e_{m+2} = \mu_m e_{m+2}$, as required. Note that $1 + \mu_{m-1} = 0$ can only occur when $\lambda = 0$ and $m = 1 \pmod 3$, or when $\lambda = -1$ and $m = 0 \pmod 3$. So the uniqueness of $L(\lambda)$ is established except in the cases when $\lambda = 0$ and $\lambda = -1$. We deal with these two cases separately.

7.1. **The case $\lambda = 0$.** Let L be an $\mathbb{N}$-graded Lie algebra spanned by $\{e_i \,|\, i = 1, 2, \dots\}$, with $[e_i, e_1] = e_{i+1}$ for $i > 1$. Let $[e_3, e_2] = e_5$, $[e_4, e_2] = e_6$, $[e_5, e_2] = 0$. As above, we suppose that for some $n \geq 1$ we have $[e_{3k+1}, e_2] = e_{3k+3}$, $[e_{3k+2}, e_2] = 0$, $[e_{3k+3}, e_2] = -e_{3k+5}$ for $1 \leq k \leq n$, and we suppose that $[e_{3n+4}, e_2] = \mu e_{3n+6}$ for some $\mu \neq 1$. As above, we may assume that n is odd. We prove that $L(0)$ is the unique infinite dimensional algebra over $\mathbf{F}$ of type 2 satisfying $[e_3, e_2] = e_5$, $[e_5, e_2] = 0$ by showing that this implies that L is nilpotent.

First note that

$$\begin{aligned}
0 &= [e_{3n+2}, [e_2, e_1^3]] + [e_{3n+2}, [e_1, e_2, e_2]] \\
&= [e_{3n+2}, e_2, e_1^3] - [e_{3n+2}, e_1^3, e_2] + \\
&\quad + [e_{3n+2}, e_1, e_2, e_2] + [e_{3n+2}, e_2, e_1, e_2] + [e_{3n+2}, e_2, e_2, e_1] \\
&= [e_{3n+5}, e_2]
\end{aligned}$$

We also have $[e_2, e_1^3, e_2] = 0$ which implies that $[e_{3n+1}, [e_2, e_1^3, e_2]] = 0$. This gives $(1+\mu)[e_{3n+6}, e_2] = e_{3n+8}$. If $\mu = -1$ then we have $e_{3n+8} = 0$ and L is nilpotent (as claimed). So we assume that $\mu \neq -1$ and that $[e_{3n+6}, e_2] = \frac{1}{1+\mu} e_{3n+8}$. Next,

$$[e_{3n+4}, [e_2, e_1, e_2]] = [e_{3n+4}, [e_2, e_1^3]]$$

implies that $[e_{3n+7}, e_2] = \frac{\mu}{1+\mu} e_{3n+9}$. And since $[e_4, e_2] = e_6 = [e_3, e_1^3]$ we have

$$[e_{3n+3}, [e_4, e_2]] = [e_{3n+3}, [e_3, e_1^3]],$$

which gives

$$\mu + \frac{1}{1+\mu} = -1 - \mu - \frac{1}{1+\mu} + \frac{\mu}{1+\mu}.$$

But this implies that $\mu = 0$, and so $[e_{3n+4}, e_2] = [e_{3n+5}, e_2] = 0$, $[e_{3n+6}, e_2] = e_{3n+8}$, and $[e_{3n+7}, e_2] = 0$.

Now let $m = \frac{n+1}{2}$. Then

$$\begin{aligned}
0 &= [[e_3, e_1^{3m}], [e_3, e_1^{3m}]] \\
&= \sum_{r=0}^{m} (-1)^r \binom{m}{r} [e_{3m+3}, e_1^{3r}, e_3, e_1^{3(m-r)}] \\
&= \sum_{r=0}^{m} (-1)^r \binom{m}{r} [e_{3m+3+3r}, e_3, e_1^{3(m-r)}].
\end{aligned}$$

Our inductive hypothesis implies that $[e_{3k}, e_3] = e_{3k+3}$ for $m+1 \leq k \leq n$. And $[e_{3n+3}, e_3] = -e_{3n+6}$, $[e_{3n+6}, e_3] = e_{3n+9}$. So this equation gives $me_{3n+9} = 0$. It follows that $m = 0 \pmod 3$, and hence that $n = -1 \pmod 3$.

Since $n = -1 \pmod 3$, $n > 1$, and so $[e_7, e_2] = e_9$. Hence

$$[e_{3n+1}, [e_7, e_2]] = [e_{3n+1}, e_9].$$

Since $e_9 = [e_3, e_1^3, e_1^3]$ we see that

$$\begin{aligned}
[e_{3n+1}, e_9] &= [e_{3n+1}, e_3, e_1^3, e_1^3] + [e_{3n+1}, e_1^3, e_3, e_1^3] + [e_{3n+1}, e_1^3, e_1^3, e_3] \\
&= e_{3n+10} - [e_{3n+8}, e_2].
\end{aligned}$$

And since $[e_7, e_2] = [e_4, e_1^3, e_2]$ we have

$$\begin{aligned}
[e_{3n+1}, [e_7, e_2]] &= [e_{3n+1}, e_4, e_1^3, e_2] - [e_{3n+1}, e_1^3, e_4, e_2] \\
&\quad - [e_{3n+1}, e_2, e_4, e_1^3] + [e_{3n+1}, e_2, e_1^3, e_4] \\
&= -e_{3n+10}.
\end{aligned}$$

So $[e_{3n+8}, e_2] = -e_{3n+10}$.

To summarize, we may assume that n is odd and $n = -1 \pmod 3$, and that

- $[e_{3k}, e_2] = -e_{3k+2}$, $[e_{3k+1}, e_2] = e_{3k+3}$, $[e_{3k+2}, e_2] = 0$ for $2 \leq k \leq n$,
- $[e_{3n+3}, e_2] = -e_{3n+5}$, $[e_{3n+4}, e_2] = [e_{3n+5}, e_2] = 0$,
- $[e_{3n+6}, e_2] = e_{3n+8}$, $[e_{3n+7}, e_2] = 0$, $[e_{3n+8}, e_2] = -e_{3n+10}$.

We let $n = 2cq - 1$, where q is a power of 3 and where c is coprime to 3. Then we make a further inductive assumption that for some t with $n + 2 \leq t \leq n + q$ we have

- $[e_{3k}, e_2] = e_{3k+2}$, $[e_{3k+1}, e_2] = 0$, $[e_{3k+2}, e_2] = -e_{3k+4}$ for $n + 2 \leq k \leq t$.

We show that this implies that $[e_{3t+3}, e_2] = e_{3t+5}$, $[e_{3t+4}, e_2] = 0$, $[e_{3t+5}, e_2] = -e_{3t+7}$. We have to divide the proof that $[e_{3t+3}, e_2] = e_{3t+5}$ into two cases depending on whether t is odd or even.

If t is odd let $m = \frac{t-1}{2}$. Then, since $e_{3m+4} = [e_4, e_1^{3m}]$, we see that the equation $[e_{3m+4}, e_{3m+4}] = 0$ gives

$$\sum_{r=0}^{m} (-1)^r \binom{m}{r} [e_{3m+4}, e_1^{3r}, e_4, e_1^{3(m-r)}] = 0.$$

Now $[e_{3k+1}, e_4] = 0$ for $m < k \leq n$ and for $n + 1 < k < t$, $[e_{3n+4}, e_4] = e_{3n+8}$, $[e_{3t+1}, e_4] = -e_{3t+5} + [e_{3t+3}, e_2]$. So we obtain

$$(-1)^{n-m} \binom{m}{n-m} e_{3t+5} - (-1)^m e_{3t+5} + (-1)^m [e_{3t+3}, e_2] = 0,$$

which implies that

$$[e_{3t+3}, e_2] = (1 + \binom{m}{n-m}) e_{3t+5}.$$

Now we can write $t = n + 2s$ for some s with $2 \leq 2s < q$. So

$$\binom{m}{n-m} = \binom{m}{2m-n} = \binom{cq+s-1}{2s-1} = 0 \pmod{3},$$

and $[e_{3t+3}, e_2] = e_{3t+5}$.

Now consider the case when t is even. We have $e_{3(t-n)} = [e_{3(t-n)-2}, e_2]$, and so

$$[e_{3n+5}, [e_{3(t-n)-2}, e_2]] = [e_{3n+5}, e_{3(t-n)}].$$

Now

$$\begin{aligned} [e_{3n+5}, e_{3(t-n)}] &= [e_{3n+5}, [e_3, e_1^{3(t-n-1)}]] \\ &= \sum_{r=0}^{t-n-1} (-1)^r \binom{t-n-1}{r} [e_{3n+5}, e_1^{3r}, e_3, e_1^{3(t-n-1-r)}]. \end{aligned}$$

Since $[e_{3n+5}, e_3] = -e_{3n+8}$, $[e_{3k+5}, e_3] = e_{3k+8}$ for $n < k < t - 1$, $[e_{3t+2}, e_3] = -e_{3t+5} - [e_{3t+3}, e_2]$, and since $t - n - 1$ is even, this implies that

$$[e_{3n+5}, e_{3(t-n)}] = -e_{3t+5} - [e_{3t+3}, e_2].$$

Also

$$\begin{aligned} [e_{3n+5}, [e_{3(t-n)-2}, e_2]] &= [e_{3n+5}, e_{3(t-n)-2}, e_2] \\ &= [e_{3n+5}, [e_4, e_1^{3(t-n-2)}], e_2]. \end{aligned}$$

Since $[e_{3n+5}, e_4] = e_{3n+9}$ and $[e_{3k+5}, e_4] = 0$ for $n < k \leq t-2$, this implies that

$$[e_{3n+5}, [e_{3(t-n)-2}, e_2]] = [e_{3t+3}, e_2].$$

So the equation

$$[e_{3n+5}, [e_{3(t-n)-2}, e_2]] = [e_{3n+5}, e_{3(t-n)}]$$

implies that $[e_{3t+3}, e_2] = e_{3t+5}$.

So $[e_{3t+3}, e_2] = e_{3t+5}$ whether t is odd or even.

Next note that the equations

$$\begin{aligned} [e_{3t+1}, [e_2, e_1, e_2]] &= [e_{3t+1}, [e_2, e_1^3]], \\ [e_{3t+2}, [e_2, e_1, e_2]] &= [e_{3t+2}, [e_2, e_1^3]] \end{aligned}$$

give $[e_{3t+4}, e_2] = 0$, $[e_{3t+5}, e_2] = -e_{3t+7}$.

So, by induction, we may assume that

- $[e_{3k}, e_2] = e_{3k+2}$, $[e_{3k+1}, e_2] = 0$, $[e_{3k+2}, e_2] = -e_{3k+4}$ for $n+2 \leq k \leq n+q+1$.

Finally, let $m = \frac{n+q}{2}$. We have

$$\begin{aligned} 0 &= [[e_3, e_1^{3m}], [e_3, e_1^{3m}]] \\ &= \sum_{r=0}^{m} (-1)^r \binom{m}{r} [e_{3m+3}, e_1^{3r}, e_3, e_1^{3(m-r)}]. \end{aligned}$$

We have $[e_{3k}, e_3] = e_{3k+3}$ for $m+1 \leq k \leq n$ and for $n+1 < k \leq n+q+1$, $[e_{3n+3}, e_3] = -e_{3n+6}$. Since $\sum_{r=0}^{m}(-1)^r\binom{m}{r} = 0$, we obtain $\binom{m}{q} e_{3n+3q+6} = 0$. Since $m = cq + \frac{q-1}{2}$, $\binom{m}{q} = c \neq 0 \pmod 3$, and so $e_{3n+3q+6} = 0$.

Thus the assumption that $[e_{3n+4}, e_2] \neq e_{3n+6}$ implies that L is nilpotent in every case. This completes our analysis of the case when $\lambda = 0$.

7.2. **The case $\lambda = -1$.** Let L be an $\mathbb{N}$-graded Lie algebra spanned by $\{e_i \mid i = 1, 2, \dots\}$, with $[e_i, e_1] = e_{i+1}$ for $i > 1$. Let $[e_3, e_2] = e_5$, $[e_4, e_2] = e_6$, $[e_5, e_2] = -e_7$. Repeating the argument above, we have $[e_6, e_2] = \mu_6 e_8 = 0$ (since 6 is even), $[e_7, e_2] = \mu_7 e_9 = e_9$ (since $7 \neq 0 \pmod 3$), and $[e_8, e_2] = \mu_8 e_{10} = -e_{10}$ (since 8 is even). And we may suppose that for some even $n \geq 2$ we have $[e_{3k}, e_2] = 0$, $[e_{3k+1}, e_2] = e_{3k+3}$, $[e_{3k+2}, e_2] = -e_{3k+4}$ for $2 \leq k \leq n$, and that $[e_{3n+3}, e_2] = \mu e_{3n+6}$ for some $\mu \neq 0$. We prove that $L(-1)$ is the unique infinite dimensional algebra over $\mathbf{F}$ of type 2 satisfying $[e_3, e_2] = e_5$, $[e_5, e_2] = -e_7$ by showing that this implies that L is nilpotent.

The relation

$$[e_{3n+1}, [e_2, e_1, e_2]] = [e_{3n+1}, [e_2, e_1^3]]$$

gives $[e_{3n+4}, e_2] = (1+\mu) e_{3n+6}$. And the relation

$$[e_{3n+2}, [e_2, e_1, e_2]] = [e_{3n+2}, [e_2, e_1^3]]$$

gives

$$-(1+\mu)[e_{3n+5}, e_2] = (1-\mu) e_{3n+7}.$$

If $\mu = -1$ then this gives $e_{3n+7} = 0$, and L is nilpotent. So we may suppose that $\mu \neq -1$, and that $[e_{3n+5}, e_2] = \frac{\mu-1}{\mu+1} e_{3n+7}$.

Since $[e_4, e_2] = e_6 = [e_3, e_1^3]$ we obtain

$$[e_{3n+2}, [e_4, e_2]] = [e_{3n+2}, [e_3, e_1^3]].$$

This gives

$$(1-\mu)[e_{3n+6}, e_2] + (\frac{\mu-1}{\mu+1} + \mu + 1)e_{3n+8} = [e_{3n+6}, e_2] - (\frac{\mu-1}{\mu+1} + \mu + 1)e_{3n+8}.$$

Since $\mu \neq 0$ we have $[e_{3n+6}, e_2] = -\frac{\mu}{\mu+1}e_{3n+8}$.

Let $n = 2m$. Then, since $e_{3m+4} = [e_4, e_1^{3m}]$, the equation $[e_{3m+4}, e_{3m+4}] = 0$ gives

$$\sum_{r=1}^{m}(-1)^m\binom{m}{r}[e_{3m+4}, e_1^{3r}, e_4, e_1^{3(m-r)}] = 0.$$

Now $[e_{3k+1}, e_4] = 0$ for $1 \leq k < n$, and $[e_{3n+1}, e_4] = \mu e_{3n+5}$, $[e_{3n+4}, e_4] = \frac{\mu(\mu-1)}{\mu+1}$. So we obtain

$$(m\mu - \frac{\mu(\mu-1)}{\mu+1})e_{3n+8} = 0.$$

So either $e_{3n+8} = 0$ (and L is nilpotent), or $m\mu(\mu+1) = \mu(\mu-1)$. But since $\mu \neq 0$, the only solution of $m\mu(\mu+1) = \mu(\mu-1)$ is $\mu = 1$ and $m = 0 \pmod 3$.

So we may assume that $n = 2cq$ where q is a power of 3 and where c is coprime to 3, and we may assume that

- $[e_3, e_2] = e_5$, $[e_4, e_2] = e_6$, $[e_5, e_2] = -e_7$,
- $[e_{3k}, e_2] = 0$, $[e_{3k+1}, e_2] = e_{3k+3}$, $[e_{3k+2}, e_2] = -e_{3k+4}$ for $2 \leq k \leq n$,
- $[e_{3n+3}, e_2] = e_{3n+5}$, $[e_{3n+4}, e_2] = -e_{3n+6}$, $[e_{3n+5}, e_2] = 0$, $[e_{3n+6}, e_2] = e_{3n+8}$.

We make the further inductive hypothesis that for some t with $n+1 \leq t < n+q$ we have

- $[e_{3k+1}, e_2] = -e_{3k+3}$, $[e_{3k+2}, e_2] = 0$, $[e_{3k+3}, e_2] = e_{3k+5}$ for $n+1 \leq k \leq t$.

We show that this implies that $[e_{3t+4}, e_2] = -e_{3t+6}$, $[e_{3t+5}, e_2] = 0$, $[e_{3t+6}, e_2] = e_{3t+8}$.

The equation

$$[e_{3t+1}, [e_2, e_1, e_2]] = [e_{3t+1}, [e_2, e_1^3]]$$

gives $[e_{3t+4}, e_2] = -e_{3t+6}$.

We have to divide the proof that $[e_{3t+5}, e_2] = 0$ into two cases depending on whether t is odd or even. First suppose that t is odd and let $m = \frac{t+1}{2}$. Then $e_{3m+2} = [e_2, e_1^{3m}]$ and so the equation $[e_{3m+2}, e_{3m+2}] = 0$ gives

$$\sum_{r=0}^{m}(-1)^r\binom{m}{r}[e_{3m+2}, e_1^{3r}, e_2, e_1^{3(m-r)}] = 0.$$

Now $[e_{3k+2}, e_2] = -e_{3k+4}$ for $m \leq k \leq n$, $[e_{3k+2}, e_2] = 0$ for $n < k \leq t$, and so we obtain

$$-\sum_{r=0}^{n-m}(-1)^r\binom{m}{r}e_{3t+7} + (-1)^m[e_{3t+5}, e_2] = 0.$$

We can write $t = n + 2s - 1 = 2cq + 2s - 1$ where $1 \le s < \frac{q+1}{2}$ so that $m = cq + s$ and $n - m = cq - s$. So

$$\sum_{r=0}^{n-m}(-1)^r\binom{m}{r} = \sum_{r=0}^{cq-s}(-1)^r\binom{cq+s}{r} = \pm\sum_{r=0}^{2s-1}(-1)^r\binom{cq+s}{r}.$$

But $2s-1 < q$ and so $\binom{cq+s}{r} = 0 \pmod 3$ for $s < r \le 2s-1$. So, working modulo 3,

$$\sum_{r=0}^{n-m}(-1)^r\binom{m}{r} = \pm\sum_{r=0}^{s}(-1)^r\binom{cq+s}{r} = \pm\sum_{r=0}^{s}(-1)^r\binom{s}{r} = 0,$$

and hence $[e_{3t+5}, e_2] = 0$.

Next suppose that t is even. The equation

$$[e_{3n+2}, [e_{3(t-n+1)}, e_2]] = 0$$

gives

$$[e_{3n+2}, e_{3(t-n+1)}, e_2] + [e_{3n+4}, e_{3(t-n+1)}] = 0.$$

Since $e_{3(t-n+1)} = [e_3, e_1^{3(t-n)}]$, this implies that

$$\sum_{r=0}^{t-n}(-1)^r\binom{t-n}{r}\left([e_{3n+2}, e_1^{3r}, e_3, e_1^{3(t-n-r)}, e_2] + [e_{3n+4}, e_1^{3r}, e_3, e_1^{3(t-n-r)}]\right) = 0.$$

Now $[e_{3n+2}, e_3] = e_{3n+5}$, $[e_{3k+2}, e_3] = -e_{3k+5}$ for $n < k \le t$, $[e_{3k+4}, e_3] = -e_{3k+7}$ for $n \le k < t$, and $[e_{3t+4}, e_3] = -e_{3t+7} - [e_{3t+5}, e_2]$. Since $t - n$ is even this equation implies that $[e_{3t+5}, e_2] = 0$.

So $[e_{3t+5}, e_2] = 0$ whether t is odd or even.

Finally

$$[e_{3t+3}, [e_2, e_1, e_2]] = [e_{3t+3}, [e_2, e_1^3]]$$

gives $[e_{3t+6}, e_2] = e_{3t+8}$. So we may assume by induction that

- $[e_{3k+1}, e_2] = -e_{3k+3}$, $[e_{3k+2}, e_2] = 0$, $[e_{3k+3}, e_2] = e_{3k+5}$ for $n + 1 \le k \le n + q$.

To complete our analysis of case 3 we let $t = \frac{n+q-1}{2}$, and we consider the equation

$$\begin{aligned} 0 &= [[e_4, e_1^{3t}], [e_4, e_1^{3t}]] \\ &= \sum_{r=0}^{m}(-1)^r\binom{t}{r}[e_{3t+4}, e_1^{3r}, e_4, e_1^{3(t-r)}]. \end{aligned}$$

Since $[e_{3k+4}, e_4] = 0$ for $t \le k < n - 1$ and for $n \le k \le 2t$, and $[e_{3n+1}, e_4] = e_{3n+5}$ this implies $\binom{t}{q}e_{3n+3q+5} = 0$. Since $t = cq + \frac{q-1}{2}$, $\binom{t}{q} = c \ne 0 \pmod 3$, and hence $e_{3n+3q+5} = 0$.

Thus the assumption that $[e_{3n+3}, e_2] \ne 0$ implies that L is nilpotent in every case. This completes our analysis of the case when $\lambda = -1$.

8. Constructing the algebra with first constituent of length q

In this section we construct the algebra L with first constituent of length q which is described in Section 6. If $q = 3$, this construction gives the algebra $L(-1)$ of Section 7.

Let p be an odd prime, and let q be a power of p. Let V be a vector space of dimension q over the field $\mathbf{F}(t)$ of rational functions over the field $\mathbf{F}$ with p elements. We grade V over the cyclic group of order q,

$$V = \langle v_0 \rangle \oplus \langle v_1 \rangle \oplus \cdots \oplus \langle v_{q-1} \rangle .$$

Consider the following endomorphisms D and E, of V.

$$E = \begin{cases} v_i \mapsto v_{i+1} & \text{if } i \ne q-1 \\ v_{q-1} \mapsto tv_0. \end{cases}$$

$$D = \begin{cases} v_0 \mapsto v_2 \\ v_{q-1} \mapsto -tv_1 \\ v_i \mapsto 0 & \text{otherwise.} \end{cases}$$

Thus E has weight 1, and D has weight 2.

We construct the Lie algebra A spanned by E and D in the endomorphism algebra of V.

Consider $[DE^{q-2}]$, which has weight $q \equiv 0 \pmod{q}$. For $0 \le j < q$ we have

$$v_j[DE^{q-2}] = \sum_{i=0}^{q-2} (-1)^i \binom{q-2}{i} v_j E^i D E^{q-2-i}.$$

If $j > 0$ then $v_j E^i D = 0$ unless $i = q - j - 1$ or $q - j$. For $i = q - j$ we have $v_j E^i = tv_0$, and thus

$$(-1)^i \binom{q-2}{i} v_j E^i D E^{q-2-i} = (-1)^{q-j} \binom{q-2}{q-j} tv_j,$$

while for $i = q - 1 - j$ we have $v_j E^i = v_{q-1}$, and thus

$$\begin{aligned} (-1)^i \binom{q-2}{i} v_j E^i D E^{q-2-i} &= -(-1)^{q-1-j} \binom{q-2}{q-1-j} tv_j \\ &= (-1)^{q-j} \binom{q-2}{q-1-j} tv_j. \end{aligned}$$

It follows that

$$v_j[DE^{q-2}] = v_j (-1)^{q-j} \binom{q-1}{q-j} tv_j = tv_j.$$

Similarly (for $0 \le i \le q-2$) we have $v_0 E^i D = 0$ unless $i = 0$, and so $v_0[DE^{q-2}] = tv_0$. So $[DE^{q-2}] = t \cdot 1$ is scalar multiplication by t. It follows that all the $[DE^i]$, for $0 \le i \le q-2$, are non-zero, and thus linearly independent over $\mathbf{F}$, as they have distinct weights $2, \ldots, q$. We claim that $[DE^i D] = 0$ for $0 \le i < q - 2$. To see this, consider the associative expansion of $[DE^i D]$, which is a linear combination

of monomials of the form $E^\alpha D E^\beta D$, $D E^\beta D E^\alpha$, with $\alpha + \beta = i$. Note that if $E^\alpha D E^\beta D$ or $D E^\beta D E^\alpha$ is a monomial which occurs in any of these expansions then $\beta < q - 3$. This is trivially true, except in the expansion of $[D E^{q-3} D]$. However in the expansion of $[D E^{q-3} D]$, $D E^{q-3} D$ appears twice, *but with opposite signs*. So it is sufficient to show that if $\beta < q - 3$ then $v_j D E^\beta D = 0$ for all j. But $v_j D = 0$ unless $j = 0$ or $q - 1$, and

$$v_0 D E^\beta D = v_{\beta+2} D = 0,$$

$$v_{q-1} D E^\beta D = -t v_{\beta+1} D = 0,$$

since $0 < \beta + 1, \beta + 2 < q - 2$.

Therefore

$$A = \left\langle\, E, [D E^i] : 0 \leq i \leq q - 2 \,\right\rangle$$

is q-dimensional.

Let us now consider the semidirect product $V \dotplus \mathrm{End}(V)$, and in it the Lie algebra L over $\mathbf{F}$ generated by

$$e_1 = E, \qquad e_2 = -\frac{1}{2t} \cdot v_1 - \frac{1}{2} D.$$

Recursively define $e_{i+1} = [e_i e_1]$, for $i \geq 2$. Note that for $2 \leq i \leq q$ we have by induction

$$e_i = -\frac{1}{2t} v_{i-1} - \frac{1}{2} [D E^{i-2}].$$

In particular for $i = q$ we have

$$e_q = -\frac{1}{2t} \cdot v_{q-1} - \frac{1}{2} [D E^{q-2}] = -\frac{1}{2t} v_{q-1} - \frac{t}{2} \cdot 1.$$

Therefore

$$e_{q+1} = -\frac{1}{2} v_0, \qquad e_{q+2} = -\frac{1}{2} v_1,$$

and we are in V from now on, and further commutation with e_1 and e_2 is straightforward. In particular, if $0 \leq r < q$ and $k \geq 1$, then $e_{kq+r+1} = [e_{q+1} e_1^{(k-1)q+r}] = -\frac{1}{2} t^{k-1} v_r$, and

$$[e_{kq+r+1}, e_2] = \frac{1}{4} t^{k-1} v_r D = \begin{cases} 0 \text{ unless } r = 0 \text{ or } q - 1 \\ -\frac{1}{2} e_{kq+3} \text{ if } r = 0 \\ \frac{1}{2} e_{(k+1)q+2} \text{ if } r = q - 1. \end{cases}$$

For $2 \leq i \leq q - 1$ we have

$$[e_i e_2] = \frac{1}{4} \left(\frac{1}{t} v_{i-1} D - \frac{1}{t} v_1 [D E^{i-2}] + [D E^{i-2} D] \right) = 0,$$

because of the above, and the easy fact that $v_1 [D E^{i-2}] = 0$. And

$$[e_q e_2] = \frac{1}{4} \left(\frac{1}{t} v_{q-1} D - \frac{1}{t} v_1 [D E^{q-2}] + [D E^{q-2} D] \right) = -\frac{1}{2} v_1 = e_{q+2}.$$

So L is of maximal class, graded as we want it to be. We have seen that the first constituent has length q, and that $[e_q e_2] = e_{q+2}$, $[e_{q+1}, e_2] = -\frac{1}{2} e_{q+3}$. For $n > q$ we have $[e_n, e_2] = 0$ unless n is congruent to 0 or 1 modulo q, and for $k \geq 2$ we have $[e_{kq}, e_2] = \frac{1}{2} e_{kq+2}$, $[e_{kq+1}, e_2] = -\frac{1}{2} e_{kq+3}$.

9. Constructing the extra algebras for $q = 3$

In this section we construct the algebras $L(\lambda)$ of Section 7. These are defined over a field $\mathbf{F}$ of characteristic 3, for $\lambda \in \mathbf{F}$.

The construction is similar to the one of the previous section. We rephrase it here in terms of matrices.

Let t be an indeterminate over K. Let v_1, v_2, v_3 be the standard basis of the space of row vectors $K(t)^3$. Consider the 3×3 matrices over $K(t)$

$$E = \begin{bmatrix} & 1 & \\ & & 1 \\ t & & \end{bmatrix}, \qquad D = \begin{bmatrix} & & 1 \\ \lambda t & & \\ & -(1+\lambda)t & \end{bmatrix}.$$

where as usual zero entries are omitted. We have

$$[DE] = \begin{bmatrix} (1-\lambda)t & & \\ & (1-\lambda)t & \\ & & (1-\lambda)t \end{bmatrix},$$

a scalar matrix, so that the Lie algebra spanned by D and E has dimension 3. Now consider the block 4×4 matrices

$$e_1 = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} D & 0 \\ v & 0 \end{bmatrix}.$$

Here

$$v = \frac{1}{t} v_2 = [0, \frac{1}{t}, 0] \in K(t)^3.$$

Consider the Lie algebra S spanned by e_1 and e_2. We compute

$$e_3 = [e_2 e_1] = \begin{bmatrix} [DE] & 0 \\ vE & 0 \end{bmatrix}, \qquad e_4 = [e_3 e_1] = \begin{bmatrix} 0 & 0 \\ vE^2 & 0 \end{bmatrix}.$$

Here $vE = 1/t \cdot v_3$, and $vE^2 = [1, 0, 0] = v_1$. If we define $e_{i+1} = [e_i e_1]$, for $i \geq 2$, we find thus that for $i \geq 4$ we have

$$e_i = \begin{bmatrix} 0 & 0 \\ t^j v_k & 0 \end{bmatrix},$$

where $1 \leq k \leq 3$, and $i = 3(j+1) + k$. It follows that the algebra S is infinite-dimensional over K, with basis e_i, for $i \geq 1$.

We have

$$[e_3 e_2] = \begin{bmatrix} 0 & 0 \\ v_2 & 0 \end{bmatrix} = e_5$$

and

$$[e_5 e_2] = \begin{bmatrix} 0 & 0 \\ \lambda t v_1 & 0 \end{bmatrix} = \lambda e_7.$$

It is now straightforward to see that all identities (32) are satisfied in S, so that S is isomorphic to the algebra $L(\lambda)$ of Section 7.

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